Isoperimetry and lower Ricci curvature bounds

Daniele Semola
FIM-ETH, Zürich
daniele.semola@math.ethz.ch

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General idea

I will discuss some recent results about the isoperimetric problem on *spaces* with lower Ricci curvature bounds.

We will see that for smooth Riemannian manifolds, the non compact case is much subtler than the compact one.

It also naturally leads to study the problem on more general metric measure spaces with synthetic lower Ricci curvature bounds.

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Outline

Lower Ricci bounds and GMT

$\Sigma^{N-1} \subset M^N$ is minimal and two-sided with unit normal ν. Then we can compute the second variation of the area for vector fields X such that $X = f\nu$ along Σ:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathcal{H}^{N-1}(\Phi_t(\Sigma)) = \int_\Sigma \left[|\nabla_\Sigma f|^2 - \left(|\mathrm{II}|^2 + \mathrm{Ric}(\nu,\nu)\right) f^2\right] \mathrm{d}\,\mathcal{H}^{N-1}.$$

Theorem (Simons, Ann. of Math. '68)

There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.

Corollary

There is no two-sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.

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Isoperimetric problem and isoperimetric profile

For a smooth Riemannian manifold (M, g), with volume measure vol and codimension one surface area Per, we define the isoperimetric profile $I : [0, \mathrm{vol}(M)) \to [0, +\infty)$ by

$$I(v) := \inf \{\mathrm{Per}(\Omega) \, : \, \Omega \subset M \, , \;\; \mathrm{vol}(\Omega) = v\} \; .$$

The very same definition makes sense for metric measure spaces $(X, \mathrm{d}, \mathrm{m})$. Minimizers are called isoperimetric sets/regions.

Remark

On $\mathbb{R}^N$, endowed with the Euclidean distance and the Lebesgue measure,

$$I(v) = N\omega_N^{\frac{1}{N}} v^{\frac{N-1}{N}} \; ,$$

by the classical isoperimetric inequality.

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Levy-Gromov inequality

Theorem (Gromov '86)

Let (M^N, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq N-1$. Then for any domain $\Omega \subset M$

$$\frac{\mathrm{Per}(\Omega)}{\mathrm{vol}(M)} \geq \frac{\mathrm{Per}(\Omega^*)}{\mathrm{vol}(\mathbb{S}^N)},$$

where $\Omega^* \subset \mathbb{S}^N$ is a ball such that

$$\frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(M)} = \frac{\mathrm{vol}(\Omega^*)}{\mathrm{vol}(\mathbb{S}^N)}.$$

In the original proof, the infinitesimal estimate obtained by the second variation formula is globalized to the whole manifold.

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Nonnegative Ricci curvature and Euclidean volume growth

By Bishop-Gromov, if (M^N, g) has nonnegative Ricci curvature, then the limit

$$\lim_{R\to\infty} \frac{\mathrm{vol}(B_R(p))}{\omega_N R^N} \in [0,1]$$

exists and it is independent of the base point p. We shall call it AVR, standing for Asymptotic Volume Ratio.

Theorem (See next slide)

Let (M^N, g) be complete with $\mathrm{Ric} \geq 0$. Then

$$\mathrm{Per}(E) \geq N\omega_N^{\frac{1}{N}} \mathrm{AVR}^{\frac{1}{N}} \left(\mathrm{vol}(E)\right)^{\frac{N-1}{N}},$$

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Approaches

- [Agostiniani-Fogagnolo-Mazzieri, *Invent. Math.* '20] $N = 3$, Geometric Flows;
- [Brendle, *CPAM* '20], Optimal Transport;
- [Fogagnolo-Mazzieri, *JFA* '21], $N \le 7$, Geometric Flows;
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Remark

The last three approaches cover more general ambient spaces. The most general one is in [Balogh-Kristály '21].

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Differential inequalities for the isop. profile, I

On model spaces with constant sectional curvature $K/(N-1) \in \mathbb{R}$ and dimension $N \geq 2$ the isoperimetric profile $I_{K,N}$ satisfies

$$-I''_{K,N} I_{K,N} = K + \frac{\left(I'_{K,N}\right)^2}{N-1}\,.$$

By [Bayle, *IMRN* '04] (see also [Bavard-Pansu, *ASENS* '86], [Morgan-Johnson, *IUMJ* '00] and [Ni-Wang, *JGA* '16]):

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Let (M^N, g) be a closed smooth Riemannian manifold with $\mathrm{Ric} \geq K$. Then

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The statement was previously known:

- for $N = 2$ and $K = 0$, by [Ritoré, *JGA* '02];
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Consequences

Corollary

Let (M^N, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq 0$. Then the function

$$v \mapsto \frac{I(v)}{v^{\frac{N-1}{N}}}$$

is monotone decreasing.

Corollary (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let (M^N, g) be a smooth Riemannian manifold with $\mathrm{Ric} \geq K$. Then

$$\lim_{v \to 0} \frac{(I(v))^N}{v^{N-1}} = N^N \omega_N \lim_{r \to 0} \left(\inf_{p \in M} \frac{\mathrm{vol}(B_r(p))}{v_{K/(N-1),N}(r)} \right)$$

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If we drop the compactness, the direct method of the calculus of variations fails to guarantee existence of isoperimetric regions (see [Antonelli-Glaudo '23] for a recent sharp example).

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Loss of compactness can be handled with the concentration-compactness method, [Lions, *AIHP* '84].

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Compactness and limits under lower Ricci bounds

Theorem (Gromov '82)

The class $\mathcal{M}_{N,K}$ of smooth Riemannian manifolds with $\dim \le N$ and $\mathrm{Ric} \ge K$ is precompact w.r.t. the pointed Gromov-Hausdorff topology.

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Recall the Bochner identity:

$$\frac{1}{2}\Delta|\nabla u|^2 = ||\mathrm{Hess}\, u||^2_{\mathrm{HS}} + \nabla u \cdot \nabla \Delta u + \mathrm{Ric}(\nabla u, \nabla u)\,.$$

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Examples of $\mathrm{RCD}(K, N)$ spaces are:

- Ricci limit spaces;
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Regularity of non collapsed RCD spaces

Consider an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathcal{H}^N)$. Here $\mathfrak{m} = \mathcal{H}^N$. We assume that it has no boundary for simplicity. Then:

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- The set of singular points (i.e., with non Euclidean tangent cones) might be dense [Otsu-Shioya, *JDG* '94];
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- conjecturally ([Yau '90], [Naber '20], ...), the scalar curvature is a measure. There are elementary examples where it is singular (with respect to the volume);

Area minimizing hypersurfaces

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The analogous statement holds for $d_{X\setminus E}$ on E.

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Remarks

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on $\mathrm{RCD}(K, N)$ spaces. They can be understood distributionally;
- the bounds are sharp and attained on the model spaces;
- on $\mathbb{R}^n$ the bounds imply that ∂E is a viscosity solution of the minimal surfaces equation [Savin, *Comm. PDEs* '07];
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Further remarks

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The argument for the Laplacian comparison is very flexible. It works for general solutions of minimization problems (bubbles, isoperimetric sets, ...).

Applying the Gauss-Green theorem on a tubular neighbourhood of the boundary we get sharp estimates for the first and second variation of the area.

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Thank you for your attention!