

# Isoperimetry and lower Ricci curvature bounds

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# General idea

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

I will discuss some recent results about the **isoperimetric problem** on *spaces* with **lower Ricci** curvature bounds.

We will see that for **smooth Riemannian** manifolds, the **non compact** case is much subtler than the compact one.

It also naturally leads to study the problem on more general **metric measure spaces** with **synthetic lower Ricci curvature bounds**.

# General idea

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# General idea

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Outline

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

- 1 Introduction
- 2 Main results
- 3 Ricci bounds
- 4 GMT on RCD spaces
- 5 Open questions

# Lower Ricci bounds and GMT

## GMT and lower Ricci bounds

Daniele Semola

### Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

$\Sigma^{N-1} \subset M^N$  is minimal and two-sided with unit normal  $\nu$ . Then we can compute the second variation of the area for vector fields  $X$  such that  $X = f\nu$  along  $\Sigma$ :

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{H}^{N-1}(\Phi_t(\Sigma)) = \int_{\Sigma} [|\nabla_{\Sigma} f|^2 - (|\mathbb{H}|^2 + \text{Ric}(\nu, \nu)) f^2] d\mathcal{H}^{N-1}.$$

# Lower Ricci bounds and GMT

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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*There are no closed two sided stable minimal hypersurfaces in a closed manifold with positive Ricci curvature.*



# Lower Ricci bounds and GMT

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Lower Ricci bounds and GMT

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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**Theorem (Simons, *Ann. of Math.* '68)**

*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with **positive Ricci** curvature.*

*There is no two-sided area minimizing hypersurface in a closed manifold with positive Ricci curvature.*

# Lower Ricci bounds and GMT

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Lower Ricci bounds and GMT

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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*There are no closed two sided **stable** minimal hypersurfaces in a closed manifold with **positive Ricci** curvature.*

**Corollary**

*There is no two-sided **area minimizing** hypersurface in a closed manifold with **positive Ricci** curvature.*

# Isoperimetric problem and isoperimetric profile

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

For a smooth Riemannian manifold  $(M, g)$ , with volume measure  $\text{vol}$  and codimension one surface area  $\text{Per}$ , we define the **isoperimetric profile**  $I : [0, \text{vol}(M)) \rightarrow [0, +\infty)$  by

$$I(v) := \inf \{ \text{Per}(\Omega) : \Omega \subset M, \text{vol}(\Omega) = v \} .$$

The very same definition makes sense for **metric measure spaces**  $(X, d, m)$ . Minimizers are called **isoperimetric sets/regions**.

On  $\mathbb{R}^N$ , endowed with the Euclidean distance and the Lebesgue measure,

$$I(v) = N\omega_N^{\frac{1}{N}} v^{\frac{N-1}{N}} ,$$

by the classical isoperimetric inequality.

# Isoperimetric problem and isoperimetric profile

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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On  $\mathbb{R}^N$ , endowed with the Euclidean distance and the Lebesgue measure,

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by the classical isoperimetric inequality.

# Isoperimetric problem and isoperimetric profile

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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## Remark

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$$I(v) = N\omega_N^{\frac{1}{N}} v^{\frac{N-1}{N}} ,$$

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# Levy-Gromov inequality

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq N - 1$ .  
Then for any domain  $\Omega \subset M$

$$\frac{\text{Per}(\Omega)}{\text{vol}(M)} \geq \frac{\text{Per}(\Omega^*)}{\text{vol}(S^N)},$$

where  $\Omega^* \subset S^N$  is a ball such that

$$\frac{\text{vol}(\Omega)}{\text{vol}(M)} = \frac{\text{vol}(\Omega^*)}{\text{vol}(S^N)}.$$

It is an open question whether the same inequality also holds for general RCD spaces. In particular, it is not known whether the inequality holds for general RCD spaces with boundary.



# Levy-Gromov inequality

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

## Theorem (Gromov '86)

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In the original proof, the infinitesimal estimate obtained by the second variation formula is globalized to the whole manifold.

# Levy-Gromov inequality

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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## Remark

In the original proof, the *infinitesimal* estimate obtained by the second variation formula is *globalized* to the whole manifold.

# Nonnegative Ricci curvature and Euclidean volume growth

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

By Bishop-Gromov, if  $(M^N, g)$  has nonnegative Ricci curvature, then the limit

$$\lim_{R \rightarrow \infty} \frac{\text{vol}(B_R(p))}{\omega_N R^N} \in [0, 1]$$

exists and it is independent of the base point  $p$ . We shall call it AVR, standing for Asymptotic Volume Ratio.

*Let  $(M^N, g)$  be complete with  $\text{Ric} \geq 0$ . Then*

$$\text{Per}(E) \geq N \omega_N^{\frac{1}{N}} \text{AVR}^{\frac{1}{N}} (\text{vol}(E))^{\frac{N-1}{N}},$$

*for any Borel set  $E \subset M$ .*

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GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

By **Bishop-Gromov**, if  $(M^N, g)$  has **nonnegative Ricci** curvature, then the limit

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GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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**Theorem** (See next slide)

*Let  $(M^N, g)$  be complete with  $\text{Ric} \geq 0$ . Then*

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# Approaches

## GMT and lower Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD spaces

Open questions

• [Agostiniani-Fogagnolo-Mazzieri, *Invent. Math.* '20]  $N \leq 3$ , Geometric Flows;

• [Brendle, *CPAM* '20], Optimal Transport;

• [Agostiniani-Fogagnolo-Mazzieri, *Int. Math. Res. Not.* '20], Geometric Flows;

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# Approaches

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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- [Brendle, *CPAM* '20], Optimal Transport;
- [Fogagnolo-Mazzieri, *JFA* '21],  $N \leq 7$ , Geometric Flows;
- [Balogh-Kristály, *Math. Ann.* '21], Brunn-Minkowski;
- [Antonelli-Pasqualetto-Pozzetta-S., '22];
- [Cavalletti-Manini, '22], Localization technique.

## Remark

The last three approaches cover more general ambient spaces.  
The most general one is in [Balogh-Kristály '21].



# Approaches

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Approaches

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Approaches

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Approaches

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Approaches

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Approaches

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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## Remark

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# Differential inequalities for the isop. profile, I

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

On model spaces with constant sectional curvature  $K/(N-1) \in \mathbb{R}$  and dimension  $N \geq 2$  the isoperimetric profile  $I_{K,N}$  satisfies

$$-I''_{K,N} I_{K,N} = K + \frac{(I'_{K,N})^2}{N-1}.$$

By [Bayle, *IMRN* '04] (see also [Bavard-Pansu, *ASENS* '86], [Morgan-Johnson, *IUMJ* '00] and [Ni-Wang, *JGA* '16]):

Let  $(M^N, g)$  be a closed smooth Riemannian manifold with  $\text{Ric} \geq K$ . Then

$$-I'' \geq K + \frac{(I')^2}{N-1}$$

in the sense of distributions on  $(0, \text{vol}(V))$ .

# Differential inequalities for the isop. profile, I

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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By [Ba~~yle~~, *IMRN* '04] (see also [Bavard-Pansu, *ASENS* '86], [Morgan-Johnson, *IUMJ* '00] and [Ni-Wang, *JGA* '16]):

## Theorem

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in the sense of distributions on  $(0, \text{vol}(V))$ .

# The non compact case

## GMT and lower Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD spaces

Open questions

Let  $(M^N, g)$  be a complete smooth Riemannian manifold with  $\text{Ric} \geq K$ . Then

$$-V' \geq K + \frac{(V')^2}{N-1}$$

in the sense of distributions on  $(0, \text{vol}(V))$ .

The statement was previously known:

• for  $N=2$  and  $K=1$ , by Bishop (1961)

• in fact, from Bishop's inequality assumption, by

• Bishop (1961)

# The non compact case

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

Theorem (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let  $(M^N, g)$  be a *complete smooth Riemannian manifold* with  $\text{Ric} \geq K$ . Then

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*in the sense of distributions on  $(0, \text{vol}(V))$ .*

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# The non compact case

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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$$-I''I \geq K + \frac{(I')^2}{N-1}$$

*in the sense of distributions on  $(0, \text{vol}(V))$ .*

The statement was previously known:

- for  $N = 2$  and  $K = 0$ , by [Ritoré, JGA '02];

# The non compact case

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

## Theorem (Antonelli-Pasqualetto-Pozzetta-S. '22)

Let  $(M^N, g)$  be a *complete smooth Riemannian manifold* with  $\text{Ric} \geq K$ . Then

$$-I''I \geq K + \frac{(I')^2}{N-1}$$

*in the sense of distributions on  $(0, \text{vol}(V))$ .*

The statement was previously known:

- for  $N = 2$  and  $K = 0$ , by [Fitoré, JGA '02];
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# The non compact case

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# The non compact case

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Consequences

## GMT and lower Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD spaces

Open questions

Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq 0$ . Then the function

$$v \mapsto \frac{I(v)}{v^{\frac{N-1}{N}}}$$

is monotone decreasing.

Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq 0$ . Then

for any  $\lambda > 0$  and  $\mu > 0$  there exists  $\delta > 0$  such that

$$\int_M \lambda \text{Ric} \leq \int_M \mu \text{Ric} + \delta \int_M \text{Ric}^2$$

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# Consequences

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

## Corollary

Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq 0$ . Then the function

$$v \mapsto \frac{I(v)}{v^{\frac{N-1}{N}}}$$

is *monotone decreasing*.

Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq K$ . Then

$$\lim_{r \rightarrow 0} \frac{(I(v))^{N-1}}{v^{N-1}} = N^{N-1} \omega_N \lim_{r \rightarrow 0} \left( \inf_{p \in M} \frac{\text{vol}(B_r(p))}{v_{K/(N-1), N}(r)} \right)^{\frac{1}{N-1}},$$

where  $v_{K/(N-1), N}(r)$  is the volume of the ball of radius  $r$  in the model space of dimension  $N$  and sectional curvature  $K/(N-1)$ .

# Consequences

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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Let  $(M^N, g)$  be a smooth Riemannian manifold with  $\text{Ric} \geq K$ . Then

$$\lim_{r \rightarrow 0} \frac{(I(v))^N}{v^{N-1}} = N^N \omega_N \lim_{r \rightarrow 0} \left( \inf_{p \in M} \frac{\text{vol}(B_r(p))}{v_{K/(N-1), N}(r)} \right)^2,$$

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# Consequences

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Generalized existence, I

## GMT and lower Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD spaces

Open questions

If we drop the compactness, the **direct method** of the calculus of variations fails to guarantee **existence of isoperimetric regions** (see [**Antonelli-Glaudo '23**] for a recent sharp example).

Loss of compactness can be handled with the **concentration-compactness method**, [**Lions, AHP '84**].

# Generalized existence, I

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Generalized existence, I

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Generalized existence, II

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

**Main results**

Ricci bounds

GMT on RCD  
spaces

Open questions

[Ritoré-Rosales, *TAMS* '04]: for a minimizing sequence for volume  $v$ , part of the mass converges to an isoperimetric region with volume  $\leq v$ , the remaining part diverges to infinity.

[Nardulli, *Asian J. Math* '14]: if the geometry at infinity is “uniformly bounded”, the escaping parts converge to isoperimetric regions in some pointed limits at infinity, that are smooth Riemannian manifolds too.

# Generalized existence, II

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Generalized existence, II

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Compactness and limits under lower Ricci bounds

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

*The class  $\mathcal{M}_{N,K}$  of smooth Riemannian manifolds with  $\dim \leq N$  and  $\text{Ric} \geq K$  is precompact w.r.t. the pointed Gromov-Hausdorff topology.*

*Understanding of  $\mathcal{M}_{N,K}$  is tightly linked with understanding of its limits (with respect to) the Gromov-Hausdorff topology.*

Several contributions to the theory of Ricci limit spaces by Fukaya, Anderson, Cheeger, Colding, Tian, Naber, Sormani, Wei, Kapovitch, Wilking, Jiang, ....

# Compactness and limits under lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

## Theorem (Gromov '82)

*The class  $\mathcal{M}_{N,K}$  of smooth Riemannian manifolds with  $\dim \leq N$  and  $\text{Ric} \geq K$  is precompact w.r.t. the pointed Gromov-Hausdorff topology.*

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# Compactness and limits under lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Compactness and limits under lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Compactness and limits under lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Compactness and limits under lower Ricci bounds

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Generalized existence, III

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

*Given a complete Riemannian manifold  $(M^n, g)$  with  $\text{Ric} \geq K$  and  $\text{vol}(B_1(x)) > v_0 > 0$  for every  $x \in X$ , for any  $v > 0$  there exist:*



# Generalized existence, III

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

Theorem (Antonelli-Nardulli-Pozzetta, *ESAIM: COCV*'22)

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- a finite collection of  $M \in \mathbb{N}$  **Ricci limit spaces**  
 $(X_i, d_i) = \lim_{k \rightarrow \infty} (M, d_g, p_i^k)$  in the measured pGH topology,  
where  $d_g(p_i^k, p) \rightarrow \infty$  as  $k \rightarrow \infty$ ;
- **isop. regions**  $\Omega_0, \Omega_1, \dots, \Omega_i$  with  $\Omega_0 \subset M, \Omega_i \subset X_i$  such that:

# Generalized existence, III

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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- *isop. regions*  $\Omega_0, \Omega_1, \dots, \Omega_j$  with  $\Omega_0 \subset M, \Omega_i \subset X_i$  such that:

*with the following properties:*

$$\sum_{i=0}^j \chi^N(\Omega_i) = v.$$

# Generalized existence, III

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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  - i) there is *no loss of mass*:

$$\sum_{i=0}^M \mathcal{H}^N(\Omega_i) = v;$$

- ii) the value of the *isoperimetric profile* (of  $M$ ) is achieved:

$$\sum_{i=0}^M \text{Per}_i(\Omega_i) = I(v).$$

# Generalized existence, III

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Generalized existence, III

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Questions

**GMT and lower  
Ricci bounds**

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD  
spaces

Open questions



# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

## Question

Do **area minimizing** hypersurfaces in non smooth spaces with lower curvature bounds have **vanishing mean curvature**?

Are isoperimetric surfaces **CMC**?

# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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## Motivations

- Understand **Curvature**, (cf. with **[Gromov '19]**);
- GMT on **singular spaces** as a new tool for **classical GMT**.

# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Remarks

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

It seems worthwhile to develop a theory independent of the existence of smooth approximating sequences.

It would be interesting to know if the lower Ricci bound is preserved by subsequences.

# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD  
spaces

Open questions

## Remark

It seems worthwhile to develop a theory **independent of** the existence of **smooth** approximating sequences.

Smoothness gets lost, but the lower Ricci curvature bound is stable, in suitable sense.



# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD  
spaces

Open questions

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## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

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# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

RCD( $K, N$ ) metric measure spaces  $(X, d, m)$  are “Riemannian” spaces with Ricci bounded from below by  $K \in \mathbb{R}$ , dimension bounded above by  $1 \leq N < \infty$ .

Recall the Bochner identity:

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\text{Hess } u\|_{\text{HS}}^2 + \nabla u \cdot \nabla \Delta u + \text{Ric}(\nabla u, \nabla u).$$

# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

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A m.m.s.  $(X, d, m)$  is RCD( $K, N$ ) if:

# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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## Definition

A m.m.s.  $(X, d, m)$  is RCD( $K, N$ ) if:

- $W^{1,2}$  is a Hilbert space and functions with bounded gradient are Lipschitz;
- for sufficiently many test functions  $u : X \rightarrow \mathbb{R}$ ,

$$\frac{1}{2} \Delta |\nabla u|^2 \geq \frac{(\Delta u)^2}{N} + \nabla u \cdot \nabla \Delta u + K |\nabla u|^2.$$

# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

RCD( $K, N$ ) metric measure spaces  $(X, d, m)$  are “Riemannian” spaces with Ricci bounded from below by  $K \in \mathbb{R}$ , dimension bounded above by  $1 \leq N < \infty$ .

Recall the Bochner identity:

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\text{Hess } u\|_{\text{HS}}^2 + \nabla u \cdot \nabla \Delta u + \text{Ric}(\nabla u, \nabla u).$$

## Definition

A m.m.s.  $(X, d, m)$  is RCD( $K, N$ ) if:

- $W^{1,2}$  is a Hilbert space and functions with bounded gradient are Lipschitz;
- for sufficiently many test functions  $u : X \rightarrow \mathbb{R}$ ,

$$\frac{1}{2} \Delta |\nabla u|^2 \geq \frac{(\Delta u)^2}{N} + \nabla u \cdot \nabla \Delta u + K |\nabla u|^2.$$

# RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Remarks

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

Contributions by several authors after [Sturm, *Acta Math.* '06], [Lott-Villani, *Ann. of Math.* '09], [Ambrosio-Gigli-Savaré, *Duke Math. J.* '14].

Examples of  $\text{RCD}(K, N)$  spaces are:

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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Remarks

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

Contributions by several authors after [Sturm, *Acta Math.* '06], [Lott-Villani, *Ann. of Math.* '09], [Ambrosio-Gigli-Savaré, *Duke Math. J.* '14].

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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

Contributions by several authors after [Sturm, *Acta Math.* '06], [Lott-Villani, *Ann. of Math.* '09], [Ambrosio-Gigli-Savaré, *Duke Math. J.* '14].

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# Regularity of non collapsed RCD spaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

Consider an  $\text{RCD}(K, N)$  space  $(X, d, \mathcal{H}^N)$ . Here  $m = \mathcal{H}^N$ . We assume that it has no **boundary** for simplicity. Then:

- $(X, d)$  is a complete metric space with bounded diameter
- $(X, d)$  is a doubling metric space
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# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD  
spaces

Open questions

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# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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- $(X, d)$  is bi-Hölder homeomorphic to a smooth manifold away from a set of codimension two [Kapovitch-Mondino, Geom. Topol. '18] after [Cheeger-Golding, JDG '97];

• Cheeger-Colding conjecture:  $(X, d)$  is bi-Lipschitz homeomorphic to a smooth manifold away from a set of codimension two

# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Regularity of non collapsed RCD spaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Bad news

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

**Ricci bounds**

GMT on RCD spaces

Open questions

- The set of singular points (i.e., with non Euclidean tangent cones) might be dense [Otsu-Shioya, JDG '94];
- [Golding-Naber, *Adv. Math.* '13] build examples with no singular points where the distance is not induced by  $C^\alpha$  Riemannian metrics for any  $\alpha > 0$ ;
- Conjecturally [Grove-Li, Naber, '13], there are no examples with  $\alpha > 1$ . There are counterexamples when  $\alpha < 1$  is singular (with respect to the volume);

# Bad news

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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- conjecturally ([Yau '90], [Naber '20], ...), the **scalar curvature** is a **measure**. There are elementary examples where it is **singular** (with respect to the volume);

# Bad news

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Bad news

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

We deal with sets of finite perimeter.

- Euclidean theory pioneered by Caccioppoli and De Giorgi;

- for  $E \subset \mathbb{R}^n$  smooth,  $\text{Per}(E) = \int_{\partial E} |\nu|$ .

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We recall  $F_\mu$  is the  $\mu$ -flatness measure of  $E$ .

The Euclidean theory is due to Caccioppoli and De Giorgi.

For  $E \subset \mathbb{R}^n$ ,  $\text{Per}(E) = \int_{\partial E} |\nu|$ .

# Area minimizing hypersurfaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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- for  $E \subset \mathbb{R}^N$  smooth,  $\text{Per}(E, \cdot) = \mathcal{H}^{N-1} \llcorner \partial E$ ;

- $\text{Per}(E, \cdot)$  is a Radon measure on  $\mathbb{R}^N$ ;
- $\text{Per}(E, \cdot)$  is associated to the perimeter functional  $\text{Per}(E) = \text{Per}(E, \cdot)(\mathbb{R}^N)$ ;
- $\text{Per}(E, \cdot)$  is the total variation of the characteristic function  $\chi_E$ ;

# Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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- for  $E \subset \mathbb{R}^N$  smooth,  $\text{Per}(E, \cdot) = \mathcal{H}^{N-1} \llcorner \partial E$ ;
- on metric measure spaces from **[Ambrosio, *Adv. Math.* '02]**;
- on  $\text{RCD}(K, N)$  spaces well understood (perimeter equals codimension one measure, rectifiability with well defined dimension, Gauss-Green formula) after **[Ambrosio-Bruè-S., *GAFA* '18]**, **[Bruè-Pasqualetto-S., *JEMS* '19]**.

# Area minimizing hypersurfaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Area minimizing hypersurfaces

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Area minimizing hypersurfaces

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Laplacian comparison for minimizers

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

**GMT on RCD spaces**

Open questions

For  $K \in \mathbb{R}$  and  $1 \leq N < \infty$  let

$$\mathbb{R}^+_{K,N} := -\sqrt{K(N-1)} \tan(\sqrt{K/(N-1)}x) \text{ if } K > 0;$$

$$\mathbb{R}^0_{0,N} := 0;$$

$$\mathbb{R}^-_{K,N} := -\sqrt{-K(N-1)} \tanh(\sqrt{-K/(N-1)}x) \text{ if } K < 0.$$

# Laplacian comparison for minimizers

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Laplacian comparison for minimizers

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Laplacian comparison for minimizers

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Laplacian comparison for minimizers

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Laplacian comparison for minimizers

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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## Theorem (Mondino-S. '21)

Let  $(X, d, \mathcal{H}^N)$  be an  $\text{RCD}(K, N)$  metric measure space. Let  $E \subset X$  be a set of locally finite perimeter and assume that it is a *local perimeter minimizer*. Let  $d_{\bar{E}} : X \setminus \bar{E} \rightarrow [0, \infty)$  be the *distance function* from  $\bar{E}$ . Then

$$\Delta d_{\bar{E}} \leq \tau_{K,N} \circ d_{\bar{E}} \quad \text{on } X \setminus \bar{E}.$$

*The analogous statement holds for  $d_{X \setminus E}$  on  $E$ .*

# Laplacian comparison for minimizers

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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# Laplacian comparison for minimizers

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

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## Theorem (Mondino-S. '21)

Let  $(X, d, \mathcal{H}^N)$  be an  $\text{RCD}(K, N)$  metric measure space. Let  $E \subset X$  be a set of locally finite perimeter and assume that it is a *local perimeter minimizer*. Let  $d_{\bar{E}} : X \setminus \bar{E} \rightarrow [0, \infty)$  be the *distance function* from  $\bar{E}$ . Then

$$\Delta d_{\bar{E}} \leq \tau_{K,N} \circ d_{\bar{E}} \quad \text{on } X \setminus \bar{E}.$$

The analogous statement holds for  $d_{X \setminus E}$  on  $E$ .

# Remarks

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

**GMT on RCD spaces**

Open questions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on  $\text{RCD}(K, N)$  spaces. They can be understood distributionally;
- the bounds are sharp and achieved on the model spaces;
- the bounds imply that  $\text{RCD}$  is a stronger version of the Riemannian curvature condition. Can we use this to say something about more curvatures of the new minimizing boundaries;
- in general, on fixed spaces, the bounds follow from the Ricci bound. The case of  $\text{RCD}(K, N)$  is not so clear. The smooth Riemannian case is the present setting.

# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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  - no need to talk about **mean curvature** of the area minimizing boundary;
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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

- The distance function is not smooth even on smooth Riemannian manifolds;
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# Remarks

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

- The distance function is not smooth even on smooth Riemannian manifolds;
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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on  $RCD(K, N)$  spaces. They can be understood **distributionally**;
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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

- The distance function is not smooth even on smooth Riemannian manifolds;
- the bounds make perfectly sense on  $RCD(K, N)$  spaces. They can be understood **distributionally**;
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# Remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Further remarks

GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

The argument for the Laplacian comparison is very flexible. It works for general solutions of minimization problems (bubbles, isoperimetric sets, ...).

Applying the Gauss-Green theorem on a tubular neighbourhood of the boundary we get sharp estimates for the first and second variation of the area.

In combination with the generalized isoperimetric inequality, this is the approach to prove the sharp isoperimetric estimates.

# Further remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Further remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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In combination with the generalized existence of isoperimetric regions, this is the main tool to prove the isoperimetric comparison estimates.

# Further remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Further remarks

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Open questions

## GMT and lower Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD spaces

Open questions

## Open questions

Can we obtain every function  $f : (0, \infty) \rightarrow (0, \infty)$  such that  $f^{N-1}$  is concave as the isoperimetric profile of a complete  $N$ -dim. manifold with  $\text{Ric} \geq 0$ ?

There exists a smooth Riemannian manifold  $(M^N, g)$  with  $\text{Ric} \geq 0$  and  $\text{Sec} \geq 0$  such that GMT isoperimetric profiles do not exist in  $M$  for a sequence of volumes  $v_i \rightarrow \infty$ .

By [Antonelli-Bruè-Fogagnolo-Pozzetta, *Calc. Var.* '22] this is not possible if we make the stronger assumption that  $\text{Sec} \geq 0$ .

# Open questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

## Open question

Can we obtain every function  $f : (0, \infty) \rightarrow (0, \infty)$  such that  $f^{\frac{N}{N-1}}$  is concave as the isoperimetric profile of a complete  $N$ -dim. manifold with  $\text{Ric} \geq 0$  ?

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# Open questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Open questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Open questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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# Open questions

GMT and lower  
Ricci bounds

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

Open questions

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Can we obtain every function  $f : (0, \infty) \rightarrow (0, \infty)$  such that  $f^{\frac{N}{N-1}}$  is concave as the isoperimetric profile of a complete  $N$ -dim. manifold with  $\text{Ric} \geq 0$  ?

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**GMT and lower  
Ricci bounds**

Daniele Semola

Introduction

Main results

Ricci bounds

GMT on RCD  
spaces

**Open questions**

Thank you for your attention!