# Some rigidity results for minimal graphs over unbounded domains

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*Sobolev inequalities in the Alps*

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# The Minimal Surface Equation (MSE)

Around 1762, *G.L. Lagrange* consider the following problem :

> *Find the graph of a smooth function $u = u(x, y)$, over a two-dimensional bounded and smooth domain $\Omega$, having least area among all graphs that assume given values at the boundary of $\Omega$.*

Lagrange showed that, *if such a graph exists*, then $u$ is a solution of

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \textit{in} \quad \Omega \tag{1}$$

- the graph of $u$ is called *minimal graph*,
- the above equation is named *Minimal Surface Equation (MSE)*,
- (most likely) this was the beginning of the Calculus of Variations :

(MSE) is the Euler-Lagrange equation associated with the area functional

$$\mathcal{A}(u) := \int_\Omega \sqrt{1+|\nabla u|^2}. \tag{2}$$

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# The Minimal Surface Equation (MSE)

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \qquad \mathit{in} \quad \Omega \subseteq \mathbb{R}^N,$$

- the l.h.s. of (MSE) is equal to $NH(x)$,
- $H(x)$ is the *the mean curvature* of the graph of $u$ at the point $(x, u(x)) \in \mathbb{R}^{N+1}$.
- This fact was first observed, for $N = 2$, by J.B. Meusnier in 1776.
- A *minimal graph* provides a simple and natural example of "minimal submanifold" of dimension $N$.

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# Bernstein's Theorem

**Theorem (S.N. Bernstein, 1915)**

*Let $u \in C^2(\mathbb{R}^2)$ be a solution of the minimal surface equation (MSE)*

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \textit{on} \quad \mathbb{R}^2 \tag{3}$$

*Then $u$ is an affine function, i.e.,*

$$u(x,y) = \alpha x + \beta y + \gamma,$$

*for some real constant $\alpha, \beta, \gamma$.*

The proof is based on a Liouville-type theorem for elliptic (*not uniformly elliptic*) operator, which holds true only in dimension 2.

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**Theorem (S.N. Bernstein, 1915 - E. Hopf, 1950)**

*Let $a, b, c : \mathbb{R}^2 \to \mathbb{R}$ be functions such that the symmetric matrix :*

$$\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix} \qquad \text{is positive definite for every } (x,y) \in \mathbb{R}^2.$$

*Let $u \in C^2(\mathbb{R}^2, \mathbb{R})$ be a solution of :*

$$\begin{cases} a(x,y)\frac{\partial^2 u}{\partial x^2}(x,y) + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y}(x,y) + c(x,y)\frac{\partial^2 u}{\partial y^2}(x,y) = 0 \quad \text{in } \mathbb{R}^2, \\ u(x,y) = o(\sqrt{x^2+y^2}) \qquad \text{as} \quad \sqrt{x^2+y^2} \to +\infty. \end{cases} \tag{4}$$

*Then $u$ is a constant function.*

# Proof of Bernstein's Theorem in $\mathbb{R}^2$

A direct calculation shows that the smooth functions

$$v_1 = \arctan\left(\frac{\partial u}{\partial x}\right), \qquad v_2 = \arctan\left(\frac{\partial u}{\partial y}\right)$$

are *bounded solutions*, on $\mathbb{R}^2$, of the equation :

$$\left(1+\left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + \left(1+\left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 v}{\partial y^2} = 0.$$

*An application of the previous Theorem with the matrix*

$$\begin{pmatrix} \left(1+\left(\frac{\partial u}{\partial y}\right)^2\right) & -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} & \left(1+\left(\frac{\partial u}{\partial x}\right)^2\right) \end{pmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 1 + |\nabla u|^2$$

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$v_1, v_2$ *are constant* $\Longrightarrow \nabla u = const.$

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$u$ *is an affine function.*

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# The Bernstein property (BP)

The natural problem of whether the only solutions of the minimal surface equation on $\mathbb{R}^N$, $N \geq 3$, are first degree polynomials became known as *Bernstein's problem (or Bernstein property)* .

This problem resisted for a half-century and was solved thanks to the combined efforts of some giants of mathematics of the XX century.

*The Bernstein property (BP) is true in :*

- $\mathbb{R}^3$, *E. De Giorgi (1965)*
- $\mathbb{R}^4$, *F.J. Almgren (1966)*
- $\mathbb{R}^N$, *for* $N \leq 7$, *J. Simons (1968)*

The proofs are based on the *deep connection* between minimal graphs defined over $\mathbb{R}^N$ and minimal (*area-minimizing*) cones in $\mathbb{R}^N$.

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# Failure of Bernstein property (BP)

*The Bernstein property fails in* $\mathbb{R}^N$*, for any* $N \geq 8$.

In 1969, E. Bombieri, E. De Giorgi and E. Giusti, settled Bernstein's problem proving the *existence of a non-affine solution* of the minimal surface equation (MSE) in $\mathbb{R}^N$, for any $N \geq 8$.

Their (amazing) proof relies on the existence of a minimal, and area-minimizing cone (*Simons' cone*)

$$C_{4,4} := \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \quad : \quad |x|^2 < |y|^2\}$$

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# Moser's Theorem

**Theorem (J. Moser, 1961)**

*Let $N \geq 2$ and $u \in C^2(\mathbb{R}^N)$ be a solution of :*

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{on} \quad \mathbb{R}^N, \\ \nabla u \in L^\infty(\mathbb{R}^N). \end{cases} \tag{5}$$

*Then u is an affine function.*

# Proof of Moser's Theorem

- Since $u$ is smooth, by differentiating the (MSE) we get

$$- \operatorname{div}(A(x)\nabla u_j) = 0 \qquad \text{on} \qquad \mathbb{R}^N \tag{6}$$

where $u_j$ denotes the partial derivative $\frac{\partial u}{\partial x_j}$, for any $j = 1, ..., N$, and $A = (a_{hk})$ is the real symmetric matrix whose entries are given by :

$$a_{hk} = a_{hk}(x) := \frac{\delta_{hk}}{(1+|\nabla u|^2)^{\frac{1}{2}}} - \frac{u_h u_k}{(1+|\nabla u|^2)^{\frac{3}{2}}}. \tag{7}$$

- $\lambda_{min}(A) = \frac{1}{(1+|\nabla u|^2)^{\frac{3}{2}}}$ and $\lambda_{max}(A) = \frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}}$.
- $\nabla u \in L^\infty(\mathbb{R}^N) \Longrightarrow$ equation (6) is uniformly elliptic on $\mathbb{R}^N$.
- The classic Liouville-type theorem $\Longrightarrow$ $u_j = const. \Longrightarrow$ $u$ affine.

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$$a_{hk} = a_{hk}(x) := \frac{\delta_{hk}}{(1+|\nabla u|^2)^{\frac{1}{2}}} - \frac{u_h u_k}{(1+|\nabla u|^2)^{\frac{3}{2}}}. \tag{7}$$

- $\lambda_{min}(A) = \frac{1}{(1+|\nabla u|^2)^{\frac{3}{2}}}$ and $\lambda_{max}(A) = \frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}}$.
- $\nabla u \in L^\infty(\mathbb{R}^N) \implies$ equation (6) is uniformly elliptic on $\mathbb{R}^N$.
- The classic Liouville-type theorem $\implies$ $u_j = const. \implies$ $u$ affine.

# The theorem of Bombieri and Giusti

**Theorem (E. Bombieri, E. Giusti, 1972)**

*Let $N \geq 2$ and $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{on} \quad \mathbb{R}^N,$$

*such that N-1 partial derivatives of u are bounded on $\mathbb{R}^N$.*
*Then u is an affine function.*

To prove their result, E. Bombieri and E. Giusti, demonstrate

- a new Harnack inequality for uniformly elliptic equations on minimal surfaces (oriented boundary of least area).
- if $N-1$ partial derivatives of $u$ are bounded on $\mathbb{R}^N$, then $u$ has bounded gradient on $\mathbb{R}^N$.
- Moser's Theorem $\Longrightarrow$ the Bernstein property.

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# A sharp Bernstein-type result

**Theorem (A.F., 2018)**

*Let $N \geq 8$ and $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \qquad \text{on} \quad \mathbb{R}^N,$$

*such that*

$N-7$ *partial derivatives of $u$ are bounded on one side (not necessarily the same).*

*Then $u$ is an affine function.*

The theorem is sharp.
Its proof is completely different from the one of Bombieri and Giusti.

# A rigidity result on half-spaces

**Theorem (A.F., 2022)**

*Assume $N \geq 2$ and let $\Sigma$ be an open affine half-space of $\mathbb{R}^N$. If $u \in C^2(\overline{\Sigma})$ is a solution of*

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in} \quad \Sigma, \\ u > 0 & \text{in} \quad \Sigma, \\ u = 0 & \text{in} \quad \partial\Sigma, \end{cases}$$

*then $u$ is an affine function.*

The proof is based on the following new rigidity result for entire minimal graphs.

# Rigidity on the entire space I

**Theorem (A.F., 2022)**

*Assume $N \geq 2$ and let $v \in C^2(\mathbb{R}^N)$ be a solution of the minimal surface equation*

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0 \qquad \text{on} \quad \mathbb{R}^N$$

*If for some $a, b \in \mathbb{R}$ the set $\{x \in \mathbb{R}^N : v(x) > a|x| + b\}$ is contained in an open affine half-space of $\mathbb{R}^N$, then $u$ is an affine function.*

The same conclusion remains true if the above assumption is replaced by:

*for some $a', b' \in \mathbb{R}$ the set $\{x \in \mathbb{R}^N : v(x) < a'|x| + b'\}$ is contained in an open affine half-space of $\mathbb{R}^N$.*

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# Proof of the main result on half-spaces

- W.l.o.g we may and do suppose that $\Sigma = \{x \in \mathbb{R}^N : x_N > 0\}$
- Let $v : \mathbb{R}^N \to \mathbb{R}$ be the odd extension of $u$ with respect to $\partial\Sigma$
- Thanks to the homogeneous Dirichlet boundary condition satisfied by $u$, it is easily seen that $v$ is an *entire minimal graph* such that

$$\{x \in \mathbb{R}^N : v(x) > 0\} = \{x \in \mathbb{R}^N : x_N > 0\}$$

- Theorem 1 with $a = b = 0 \Longrightarrow v$ is affine.

The latter implies that $u$ is an affine function.

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# A rigidity result on the entire space II

Another interesting consequence of Theorem 1 is the following

**Theorem II (A.F., 2022)**

Assume $N \geq 2$ and let $v \in C^2(\mathbb{R}^N)$ be a solution of the minimal surface equation

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = 0 \qquad \textit{on} \quad \mathbb{R}^N$$

If for some $a, b \in \mathbb{R}$ the set $\{x \in \mathbb{R}^N : v(x) > a|x| + b\}$ contains an open affine half-space of $\mathbb{R}^N$, then $u$ is an affine function.

The same conclusion remains true if the above assumption is replaced by:

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# Rigidity on the entire space II : proof

By assumption, the set $\{x \in \mathbb{R}^N : v(x) > a|x| + b\}$ contains an open affine half-space $\Sigma$.

Therefore, the set $\{x \in \mathbb{R}^N : v(x) < a|x| + b\}$ is contained in the open affine half-space $\Sigma' := \mathbb{R}^N \setminus \overline{\Sigma}$ and so $v$ must be an affine function thanks to Theorem 1.

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Let us recall the following well-know Liouville-type Theorem for entire minimal graphs :

**Theorem (E. Bombieri, E. De Giorgi, M.Miranda, 1969)**

*Let $v$ be an entire minimal graphs such that*

$$v(x) \geq -K(1+|x|) \qquad \forall\, x \in \mathbb{R}^N,$$

*for some $K \geq 0$. Then $v$ is an affine function.*

**Theorem (A.F., 2022)**

*Let $\Sigma$ be an open affine half-space and let $v$ be an entire minimal graphs such that*

$$v(x) \geq -K(1+|x|) \qquad \forall\, x \in \Sigma,$$

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# Rigidity on the entire space I : sketch of proof

- W.l.o.g we may and do suppose that

$$\{x \in \mathbb{R}^N : v(x) > a|x| + b\} \subset \{x \in \mathbb{R}^N : x_N > 0\} =: \Sigma$$

- $u = v - v(0)$ is again an entire solution to (MSE) with $u(0) = 0$.
- Let $U$ be the subgraph of $u$ and let $U_j$ be the one of the function $u_j$ defined by

$$u_j(x) = \frac{u(jx)}{j}, \quad x \in \mathbb{R}^N, \quad j \geq 1.$$

Since $u$ and $u_j$ are solutions of the (MSE) on $\mathbb{R}^N$, then

$U_j$ are non-trivial minimal sets of $\mathbb{R}^{N+1}$ with $0 \in \partial U_j$

Also observe that $U_j := \frac{U}{j}$.

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# Rigidity on the entire space I : sketch of proof

- By a classical *blow down* procedure, a subsequence of $U_j$ (still denoted by $U_j$) converges to a minimal cone $C \subset \mathbb{R}^{N+1}$, with vertex at the origin of $\mathbb{R}^{N+1}$ and s.t. $0 \in \partial C$.
  ($C$ is usually called a *blow down of* $U$).

  *Recall that the blow down procedure also implies :*

$$C \quad \textit{half-space} \quad \Longrightarrow \quad U \equiv C$$

- Hence, (by results of M. Miranda), $C$ is itself a subgraph of a generalized solution to the minimal surface equation $h : \mathbb{R}^N \to [-\infty, +\infty]$ and the sets

$$P = \{x \in \mathbb{R}^N \,:\, h(x) = +\infty\}, \quad N = \{x \in \mathbb{R}^N \,:\, h(x) = -\infty\}$$

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- $P = \emptyset$.

  Suppose for contradiction that $P$ is not-empty, then $P$ is a minimal cone in $\mathbb{R}^N$, with vertex at the origin of $\mathbb{R}^N$ (since $C$ is a minimal cone with vertex at the origin of $\mathbb{R}^{N+1}$).

  - *$P$ is contained in $\Sigma$.* Indeed, if $p \in P$, then there exists an integer $j > 1$ such that $u_j(p) > |v(0)| + |a||p| + |b|$ and so

$$\frac{v(jp) - v(0)}{j} > |v(0)| + |a||p| + |b| \quad \Longrightarrow \quad v(jp) > a|jp| + b,$$

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$$\Sigma \times \mathbb{R} = P \times \mathbb{R} \subset C$$

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    Therefore, $C \equiv \Sigma \times \mathbb{R}$ and so $U = C$.
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  This and the definition of $u_j$ provide the following estimate

$$\sup_{B(0,j)} u \leq \mathcal{K} j \tag{8}$$

  for some constant $\mathcal{K} > 0$.

- On the other hand, the celebrated gradient estimate of E. Bombieri, E. De Giorgi, M.Miranda, (1969) tells us that

$$\forall x \in \mathbb{R}^N,\ \forall R > 0, \quad |\nabla u(x)| \leq C_1 exp\Big[C_2\Big(\frac{\sup_{B(x,R)} u - u(x)}{R}\Big)\Big] \tag{9}$$

  where $C_1$, $C_2$ are constants depending only on the dimension $N$.

- Combining (8),(9) and letting $j \to +\infty$ we obtain that $|\nabla u| \in L^\infty(\mathbb{R}^N)$.
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- Moser's Theorem implies that $u$ is an affine function (and so is $v$).

- $P = \emptyset$ implies that the family of functions $u_j$ is equibounded from above on compact sets of $\mathbb{R}^N$.
  This and the definition of $u_j$ provide the following estimate

$$\sup_{B(0,j)} u \leq \mathcal{K} j \tag{8}$$

  for some constant $\mathcal{K} > 0$.

- On the other hand, the celebrated gradient estimate of E. Bombieri, E. De Giorgi, M.Miranda, (1969) tells us that

$$\forall\, x \in \mathbb{R}^N,\ \forall R > 0, \quad |\nabla u(x)| \leq C_1 exp\Big[C_2\Big(\frac{\sup_{B(x,R)} u - u(x)}{R}\Big)\Big] \tag{9}$$

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# Perimeter of measurable sets (De Giorgi 1954)

$X$ a Lebesgue mesurable set of $\mathbb{R}^N$,$N \geq 1$.

The *Perimeter* of $X$ in an open set $\Omega \subset \mathbb{R}^N$ is the *total variation of the distributional gradient of* $\mathbf{1}_X$ in $\Omega$, i.e.,

$$Per(X,\Omega) := \sup\left\{ \int_X divg \; : \; g \in C^1_c(\Omega,\mathbb{R}^N), \; \|g\|_\infty \leq 1 \right\} \tag{10}$$

$X$ has *locally finite perimeter* in $\Omega$ if

$$Per(X,A) < +\infty, \quad \forall \; open\,set \; A \subset\subset \Omega. \tag{11}$$

$X$ has *locally finite perimeter* in $\Omega$ if and only if $\mathbf{1}_X \in BV_{loc}(\Omega)$.

*Notation* : $\text{Per}(X,\Omega) = \int_\Omega |D\mathbf{1}_X|$.

If $X$ is smooth set of $\mathbb{R}^N$, then $\text{Per}(X,\Omega) = \mathcal{H}^{N-1}(\partial X \cap \Omega)$.

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# Minimal sets

$E \subset \mathbb{R}^N$ is a *(local) minimal set* in $\Omega$ if, for every open set $A \subset\subset \Omega$,

$$Per(E, A) < +\infty \tag{12}$$

$$Per(E, A) \leq Per(X, A), \quad \forall X \quad with \quad X \Delta E \subset A \tag{13}$$