Fractional nonlinear diffusions on manifolds: well-posedness and smoothing effects.

GABRIELE GRILLO

Dipartimento di Matematica, Politecnico di Milano
Research group "Analysis@Polimi"

Joint work with E. Berchio, M. Bonforte, M. Muratori

JUNE 29, 2023,
Workshop: "Sobolev inequalities in the Alps"

Work partially funded by PRIN project 201758MTR2

The fractional porous medium equation

We consider the following Cauchy problem, that we refer to as fractional porous medium equation (WFPME for short):

$$\begin{cases} u_t = -(-\Delta_M)^s (u^m) & \text{in } M \times (0, \infty), \\ u = u_0 & \text{on } M \times \{0\}, \end{cases} \tag{1}$$

where $s \in (0, 1), m > 1$.

The fractional porous medium equation

We consider the following Cauchy problem, that we refer to as fractional porous medium equation (WFPME for short):

$$\begin{cases} u_t = -(-\Delta_M)^s (u^m) & \text{in } M \times (0,\infty), \\ u = u_0 & \text{on } M \times \{0\}, \end{cases} \tag{1}$$

where $s \in (0,1), m > 1$. Here, M is a complete, connected, noncompact Riemannian manifold, and Δ_M the Laplace-Beltrami operator. Here, the fractional power is defined by the spectral Theorem.

The fractional porous medium equation

We consider the following Cauchy problem, that we refer to as fractional porous medium equation (WFPME for short):

$$\begin{cases} u_t = -(-\Delta_M)^s\,(u^m) & \text{in } M \times (0,\infty)\,, \\ u = u_0 & \text{on } M \times \{0\}\,, \end{cases} \tag{1}$$

where $s \in (0,1), m > 1$. Here, M is a complete, connected, noncompact Riemannian manifold, and Δ_M the Laplace-Beltrami operator. Here, the fractional power is defined by the spectral Theorem.

Our goal will be to prove basic well–posedness results for solutions, in a suitable sense, provided M satisfies appropriate geometric assumptions, and to prove smoothing effects for data in a suitable class, larger than $L^1(M)$.

The Euclidean case

When $M = \mathbb{R}^N$, equation (1) have been introduced and thoroughly studied by de Pablo, Quiros, Rodriguez, Vázquez, and then by Bonforte and Vázquez in three seminal papers: Adv. Math. 2011, CPAM 2012, Adv. Math. 2014.

The Euclidean case

When $M = \mathbb{R}^N$, equation (1) have been introduced and thoroughly studied by de Pablo, Quiros, Rodriguez, Vázquez, and then by Bonforte and Vázquez in three seminal papers: Adv. Math. 2011, CPAM 2012, Adv. Math. 2014. Among the topics dealt there with I mention (for the case $m > 1$, in fact for $m > m_c$ for an explicit $m_c < 1$:

- existence of a (strong) solution;
- conservation of mass;
- order preserving property of the evolution;
- smoothing effects, namely bounds of the form ($p \geq 1$)

$$\|u(t)\|_\infty \leq C \frac{\|u_0\|_p^{\alpha_p}}{t^{\delta_p}} \quad \forall t > 0.$$

The Euclidean case

When $M = \mathbb{R}^N$, equation (1) have been introduced and thoroughly studied by de Pablo, Quiros, Rodriguez, Vázquez, and then by Bonforte and Vázquez in three seminal papers: Adv. Math. 2011, CPAM 2012, Adv. Math. 2014. Among the topics dealt there with I mention (for the case $m > 1$, in fact for $m > m_c$ for an explicit $m_c < 1$:

- existence of a (strong) solution;
- conservation of mass;
- order preserving property of the evolution;
- smoothing effects, namely bounds of the form ($p \geq 1$)

$$\|u(t)\|_\infty \leq C\frac{\|u_0\|_p^{\alpha_p}}{t^{\delta_p}} \quad \forall t > 0.$$

Methods rely on representation formulas, i.e. on the explicit expression of the fractional laplacian in terms of a kernel, and/or on the Caffarelli-Silvestre extension method.

As such, the above representations are proper of the Euclidean setting, though extensions are possible. In our work, we shall rely on a further characterization of the fractional laplacian, meant in the spectral sense on M.

As such, the above representations are proper of the Euclidean setting, though extensions are possible. In our work, we shall rely on a further characterization of the fractional laplacian, meant in the spectral sense on M. For example one has, in fact for a large class of generators but in particular for the Laplacian on a stochastically complete manifold, and for a suitable class of functions f:

$$\begin{aligned}(-\Delta_M)^s f(x) &= c\int_0^{+\infty}[T_t f(x) - f(x)]\frac{\mathrm{d}t}{t^{1+s}}\\ &= c\int_0^{+\infty}\left(\int_M k_M(t,x,y)\,(f(y)-f(x))\,\mathrm{d}m(y)\right)\,\frac{\mathrm{d}t}{t^{1+s}},\end{aligned}$$

where m is the Riemannian measure, T_t is the heat semigroup and K_M the heat kernel on M.

As such, the above representations are proper of the Euclidean setting, though extensions are possible. In our work, we shall rely on a further characterization of the fractional laplacian, meant in the spectral sense on M. For example one has, in fact for a large class of generators but in particular for the Laplacian on a stochastically complete manifold, and for a suitable class of functions f:

$$
\begin{aligned}
(-\Delta_M)^s f(x) &= c \int_0^{+\infty} [T_t f(x) - f(x)] \frac{\mathrm{d}t}{t^{1+s}} \\
&= c \int_0^{+\infty} \left(\int_M k_M(t,x,y)\,(f(y) - f(x))\,\mathrm{d}m(y) \right) \frac{\mathrm{d}t}{t^{1+s}},
\end{aligned}
$$

where m is the Riemannian measure, T_t is the heat semigroup and K_M the heat kernel on M. In fact, the second equality holds if

$$
\int_M k_M(t,x,y)\,\mathrm{d}m(y) = 1, \quad \forall x \in M
$$

which will follow under our assumptions on M (see below).

Assumptions on M

Assumption 1 (Ricci+Faber-Krahn)

We require that M is an N-dimensional and that:

$$\mathrm{Ric}(M) \geq -(N-1)k \quad \text{for some } k > 0\,. \tag{2}$$

Assumptions on M

Assumption 1 (Ricci+Faber-Krahn)

We require that M is an N-dimensional and that:

$$\mathrm{Ric}(M) \geq -(N-1)k \quad \textit{for some } k > 0\,. \tag{2}$$

Besides, we require that $\exists c > 0$ s.t. the Faber-Krahn inequality holds:

$$\lambda_1(\Omega) \geq c\, m(\Omega)^{-\frac{2}{N}} \tag{3}$$

for any Ω is open, relatively compact, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_M$ with homogeneous Dirichlet b.c..

Assumptions on M

Assumption 1 (Ricci+Faber-Krahn)

We require that M is an N-dimensional and that:

$$\mathrm{Ric}(M) \geq -(N-1)k \quad \text{for some } k > 0\,. \tag{2}$$

Besides, we require that $\exists c > 0$ s.t. the Faber-Krahn inequality holds:

$$\lambda_1(\Omega) \geq c\, m(\Omega)^{-\frac{2}{N}} \tag{3}$$

for any Ω is open, relatively compact, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta_M$ with homogeneous Dirichlet b.c..

Notice that (3) is equivalent to the Nash inequality

$$\|f\|_2^{1+\frac{2}{N}} \leq C\,\|f\|_1^{\frac{2}{N}}\,\|\nabla f\|_2$$

or to the Sobolev inequality, if $N \geq 3$.

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x, y)$ is geodesic distance):

$$k_M(t, x, y) \le \frac{C}{t^{\frac{N}{2}}}\, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}}\ .$$

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x, y)$ is geodesic distance):

$$k_M(t, x, y) \leq \frac{C}{t^{\frac{N}{2}}}\, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}} .$$

It follows that *M* is *s*-nonparabolic, in the sense that

$$\mathbb{G}^s_M(x, y) := c \int_0^{+\infty} \frac{k_M(t, x, y)}{t^{1-s}}\,\mathrm{dt}$$

(the fractional Green function) is finite for all $x, y \in M$ with $x \neq y$.

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x,y)$ is geodesic distance):

$$k_M(t,x,y) \le \frac{C}{t^{\frac{N}{2}}}\, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}}\,.$$

It follows that *M* is *s*-nonparabolic, in the sense that

$$\mathbb{G}^s_M(x,y) := c\int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

(the fractional Green function) is finite for all $x, y \in M$ with $x \neq y$. Besides, one has the Euclidean-type bound

$$\mathbb{G}^s_M(x,y) \le \frac{C}{r(x,y)^{N-2s}} \qquad \forall x, y \in M\,,$$

but the decay of of $\mathbb{G}^s_M$ at infinity can be much faster.

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x, y)$ is geodesic distance):

$$k_M(t, x, y) \le \frac{C}{t^{\frac{N}{2}}} \, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}} .$$

It follows that M is s-nonparabolic, in the sense that

$$\mathbb{G}_M^s(x, y) := c \int_0^{+\infty} \frac{k_M(t, x, y)}{t^{1-s}} \, \mathrm{dt}$$

(the fractional Green function) is finite for all $x, y \in M$ with $x \ne y$. Besides, one has the Euclidean-type bound

$$\mathbb{G}_M^s(x, y) \le \frac{C}{r(x, y)^{N-2s}} \qquad \forall x, y \in M ,$$

but the decay of of $\mathbb{G}_M^s$ at infinity can be much faster. Finally, it can be shown that the property $\int_M k_M(t, x, y) \, \mathrm{d}m(y) = 1 \ \forall x \in M$ holds.

It is known (see e.g. Hebey 99) that (3) implies, for all $\varepsilon > 0$, $x, y \in M$ and $t > 0$, Gaussian upper bounds ($r(x,y)$ is geodesic distance):

$$k_M(t,x,y) \leq \frac{C}{t^{\frac{N}{2}}}\, e^{-\frac{r(x,y)^2}{(4+\varepsilon)t}}\,.$$

It follows that *M* is *s*-nonparabolic, in the sense that

$$\mathbb{G}_M^s(x,y) := c\int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

(the fractional Green function) is finite for all $x, y \in M$ with $x \neq y$. Besides, one has the Euclidean-type bound

$$\mathbb{G}_M^s(x,y) \leq \frac{C}{r(x,y)^{N-2s}} \qquad \forall x, y \in M\,,$$

but the decay of of $\mathbb{G}_M^s$ at infinity can be much faster. Finally, it can be shown that the property $\int_M k_M(t,x,y)\,\mathrm{d}m(y) = 1\ \forall x \in M$ holds.

We prove in Theorem 1 existence of a weak-dual solution under Assumption 1 and for a class of data larger than L^1, in Theorems 2 and 4 smoothing effects for different set of data.

Further assumptions on M

To prove stronger results, we shall sometimes use the following additional hypotheses.

Further assumptions on M

To prove stronger results, we shall sometimes use the following additional hypotheses.

Assumption 2 (Cartan-Hadamard)

We require that M is an N-dimensional Cartan-Hadamard manifold, namely that M is complete, simply connected and has everywhere nonpositive sectional curvature.

Further assumptions on M

To prove stronger results, we shall sometimes use the following additional hypotheses.

Assumption 2 (Cartan-Hadamard)

We require that M is an N-dimensional Cartan-Hadamard manifold, namely that M is complete, simply connected and has everywhere nonpositive sectional curvature.

If M is Cartan-Hadamard, the Faber-Krahn inequality is always true, but a lower Ricci bound need not be. Assumption 2 holds both in $\mathbb{R}^n$ and on hyperbolic space $\mathbb{H}^N$, the latter being the simply connected, N-dimensional manifold whose sectional curvatures are everywhere equal to -1.

Further assumptions on M

To prove stronger results, we shall sometimes use the following additional hypotheses.

Assumption 2 (Cartan-Hadamard)

We require that M is an N-dimensional Cartan-Hadamard manifold, namely that M is complete, simply connected and has everywhere nonpositive sectional curvature.

If M is Cartan-Hadamard, the Faber-Krahn inequality is always true, but a lower Ricci bound need not be. Assumption 2 holds both in $\mathbb{R}^n$ and on hyperbolic space $\mathbb{H}^N$, the latter being the simply connected, N-dimensional manifold whose sectional curvatures are everywhere equal to -1.

We prove, in Theorems 2 and 4, smoothing effects for all times and for different set of data.

In order to obtain some improved estimates for large times, we will also require a stricter hypothesis:

In order to obtain some improved estimates for large times, we will also require a stricter hypothesis:

Assumption 3 (Upper sectional)

M is Cartan-Hadamard and, besides,

$$\sec(M) \le -\kappa \qquad \textit{for a given } \kappa > 0\,.$$

In order to obtain some improved estimates for large times, we will also require a stricter hypothesis:

Assumption 3 (Upper sectional)

M is Cartan-Hadamard and, besides,

$$\sec(M) \leq -\kappa \qquad \textit{for a given } \kappa > 0\,.$$

Notice that the main example in which Assumption Upper Sectional holds is the hyperbolic space $\mathbb{H}^n$, which was the object of a specific study in Berchio, Bonforte, Ganguly, G., Calc. Var 2020.

In order to obtain some improved estimates for large times, we will also require a stricter hypothesis:

Assumption 3 (Upper sectional)

M is Cartan-Hadamard and, besides,

$$\sec(M) \leq -\kappa \qquad \textit{for a given } \kappa > 0\,.$$

Notice that the main example in which Assumption Upper Sectional holds is the hyperbolic space $\mathbb{H}^n$, which was the object of a specific study in Berchio, Bonforte, Ganguly, G., Calc. Var 2020.

We prove, in Theorem 3, smoothing effects for large times, for a class of data larger than L^1 ,the bounds being stronger than the ones given in Theorem 2, and similar to the long time behaviour proved in Vázquez, JMPA 2015 on $\mathbb{H}^N$, and to the smoothing effect by G., Muratori, Nonlin. Anal. 2016 for general manifolds satisfying Assumption 3.

On the concept of solution

Let $\mathbb{G}_M^s$ be the fractional Green function on M. We define, for every fixed $x_0 \in M$, $B_1(x_0)$ denoting the Riemannian ball centered in x_0 of radius one:

$$\|f\|_{L^1_{x_0,\mathbb{G}_M^s}} := \int_{B_1(x_0)} |f(x)|\, \mathrm{d}m(x) + \int_{M\setminus B_1(x_0)} |f(x)|\, \mathbb{G}_M^s(x,x_0)\, \mathrm{d}m(x)\,.$$

On the concept of solution

Let $\mathbb{G}^s_M$ be the fractional Green function on M. We define, for every fixed $x_0 \in M$, $B_1(x_0)$ denoting the Riemannian ball centered in x_0 of radius one:

$$\|f\|_{L^1_{x_0,\mathbb{G}^s_M}} := \int_{B_1(x_0)} |f(x)| \, \mathrm{d}m(x) + \int_{M \setminus B_1(x_0)} |f(x)| \, \mathbb{G}^s_M(x, x_0) \, \mathrm{d}m(x) \, .$$

Accordingly, we introduce the following space:

$$L^1_{\mathbb{G}^s_M}(M) := \left\{ f : M \to \mathbb{R} \text{ measurable} : \sup_{x_0 \in M} \|f\|_{L^1_{x_0,\mathbb{G}^s_M}} < +\infty \right\},$$

On the concept of solution

Let $\mathbb{G}_M^s$ be the fractional Green function on M. We define, for every fixed $x_0 \in M$, $B_1(x_0)$ denoting the Riemannian ball centered in x_0 of radius one:

$$\|f\|_{L^1_{x_0,\mathbb{G}_M^s}} := \int_{B_1(x_0)} |f(x)| \, \mathrm{d}m(x) + \int_{M \setminus B_1(x_0)} |f(x)| \, \mathbb{G}_M^s(x, x_0) \, \mathrm{d}m(x) \, .$$

Accordingly, we introduce the following space:

$$L^1_{\mathbb{G}_M^s}(M) := \left\{ f : M \to \mathbb{R} \text{ measurable} : \ \sup_{x_0 \in M} \|f\|_{L^1_{x_0,\mathbb{G}_M^s}} < +\infty \right\},$$

endowed with the norm

$$\|f\|_{L^1_{\mathbb{G}_M^s}} := \sup_{x_0 \in M} \|f\|_{L^1_{x_0,\mathbb{G}_M^s}} \, .$$

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t\left[(-\Delta_M)^{-s}u\right] + u^m = 0\,.$$

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t \left[(-\Delta_M)^{-s} u\right] + u^m = 0 \, .$$

Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) *if, for every $T > 0$:*

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t\left[(-\Delta_M)^{-s}u\right] + u^m = 0\,.$$

Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) if, for every $T > 0$:

- *$u \in C^0([0,T]; L^1_{x_0,\mathbb{G}^s_M}(M))$ for all $x_0 \in M$;*

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t\left[(-\Delta_M)^{-s}u\right] + u^m = 0\,.$$

Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) if, for every $T > 0$:

- $u \in C^0([0,T]; L^1_{x_0,\mathbb{G}^s_M}(M))$ *for all* $x_0 \in M$*;*
- $u^m \in L^1((0,T); L^1_{loc}(M))$*;*

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t\left[(-\Delta_M)^{-s}u\right] + u^m = 0\,.$$

Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) if, for every $T > 0$:

- $u \in C^0([0,T]; L^1_{x_0,\mathbb{G}^s_M}(M))$ *for all* $x_0 \in M$;
- $u^m \in L^1((0,T); L^1_{loc}(M))$;
- $$\int_0^T\int_M \partial_t\psi\,(-\Delta_M)^{-s}u\,dm\,\mathrm{d}t - \int_0^T\int_M u^m\,\psi\,dm\,\mathrm{d}t = 0$$

  for every test function $\psi \in C^1_c((0,T); L^\infty_c(M))$;

The definition of weak dual solution is based on the observation that applying the operator $(-\Delta_M)^{-s}$ to both sides of the equation we would formally obtain the "dual equation"

$$\partial_t\left[(-\Delta_M)^{-s}u\right] + u^m = 0\,.$$

Definition 1

Let $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, with $u_0 \geq 0$. We say that u is a Weak Dual Solution (WDS) to problem (1) *if, for every $T > 0$:*

- $u \in C^0([0,T]; L^1_{x_0,\mathbb{G}^s_M}(M))$ *for all* $x_0 \in M$;
- $u^m \in L^1((0,T); L^1_{loc}(M))$;
- $$\int_0^T \int_M \partial_t\psi\,(-\Delta_M)^{-s}u\,dm\,\mathrm{d}t - \int_0^T \int_M u^m\,\psi\,dm\,\mathrm{d}t = 0$$
  for every test function $\psi \in C^1_c((0,T); L^\infty_c(M))$;
- $u(0,\cdot) = u_0$ *a.e. in* M.

On the class of data

The following results are taken from Berchio, Bonforte, G., Muratori, preprint 2022.

On the class of data

The following results are taken from Berchio, Bonforte, G., Muratori, preprint 2022.

Under Assumption 1, it clearly holds $L^1(M) \subseteq L^1_{\mathbb{G}^s_M}(M)$, and $L^1_{\mathbb{G}^s_M}(M) \subseteq L^1_{x_0,\mathbb{G}^s_M}(M)$ is clear by definition. One may then wonder whether those spaces actually coincide. The answer is negative. In fact we prove what follows:

On the class of data

The following results are taken from Berchio, Bonforte, G., Muratori, preprint 2022.

Under Assumption 1, it clearly holds $L^1(M) \subseteq L^1_{\mathbb{G}^s_M}(M)$, and $L^1_{\mathbb{G}^s_M}(M) \subseteq L^1_{x_0,\mathbb{G}^s_M}(M)$ is clear by definition. One may then wonder whether those spaces actually coincide. The answer is negative. In fact we prove what follows:

Proposition

Let M satisfy Assumption 1. Then one has:

$$L^1(M) \subsetneq L^1_{\mathbb{G}^s_M}(M) \subsetneq L^1_{x_0,\mathbb{G}^s_M}(M) \quad \textit{for all } x_0 \in M,$$

with strict inclusions.

On the class of data

The following results are taken from Berchio, Bonforte, G., Muratori, preprint 2022.

Under Assumption 1, it clearly holds $L^1(M) \subseteq L^1_{\mathbb{G}^s_M}(M)$, and $L^1_{\mathbb{G}^s_M}(M) \subseteq L^1_{x_0,\mathbb{G}^s_M}(M)$ is clear by definition. One may then wonder whether those spaces actually coincide. The answer is negative. In fact we prove what follows:

Proposition

Let M satisfy Assumption 1. Then one has:

$$L^1(M) \subsetneq L^1_{\mathbb{G}^s_M}(M) \subsetneq L^1_{x_0,\mathbb{G}^s_M}(M) \quad \textit{for all } x_0 \in M,$$

with strict inclusions.

The result is proven by providing explicit functions which belong to one space but not the other ones.

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

Proposition

Let either $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, and let $u_0 \in L^\infty(M)$. Then, sufficient conditions for u_0 to belong to $L^1_{\mathbb{G}^s_M}(M)$ are the following:

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

Proposition

Let either $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, and let $u_0 \in L^\infty(M)$. Then, sufficient conditions for u_0 to belong to $L^1_{\mathbb{G}^s_M}(M)$ are the following:

- $M = \mathbb{R}^N$ *and* $|u_0(x)| \le \dfrac{C}{|x|^a}$ *for all* $|x| \ge R$, *for some* $a > 2s$;

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

Proposition

Let either $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, and let $u_0 \in L^\infty(M)$. Then, sufficient conditions for u_0 to belong to $L^1_{\mathbb{G}^s_M}(M)$ are the following:

- *$M = \mathbb{R}^N$ and $|u_0(x)| \leq \dfrac{C}{|x|^a}$ for all $|x| \geq R$, for some $a > 2s$;*
- *$M = \mathbb{H}^N$ and $|u_0(x)| \leq \dfrac{C}{(r(x,o))^a}$ for all $r(x,o) \geq R$, for some $a > s$.*

In order to highlight the admissible decay rate for the kind of initial data we deal with, we provide some sufficient conditions for a function to belong to $L^1_{\mathbb{G}^s_M}(M)$ in the very special cases of $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$.

Proposition

Let either $M = \mathbb{R}^N$ or $M = \mathbb{H}^N$, and let $u_0 \in L^\infty(M)$. Then, sufficient conditions for u_0 to belong to $L^1_{\mathbb{G}^s_M}(M)$ are the following:

- $M = \mathbb{R}^N$ *and* $|u_0(x)| \le \dfrac{C}{|x|^a}$ *for all* $|x| \ge R$*, for some* $a > 2s$*;*
- $M = \mathbb{H}^N$ *and* $|u_0(x)| \le \dfrac{C}{(r(x,o))^a}$ *for all* $r(x,o) \ge R$*, for some* $a > s$*.*

In both cases, initial data are allowed to decay qualitatively quite slower than functions in $L^1(M)$: the requested bound is dimension independent when $M = \mathbb{R}^N$, whereas functions in $L^1(\mathbb{H}^N)$ are expected to decay faster than $e^{-r(x,o)(N-1)}$.

Main results

Let us now state the results mentioned above.

Main results

Let us now state the results mentioned above.

Theorem 1 (Existence of a WDS for data in $L^1_{\mathbb{G}^s_M}$)

Let M satisfy Assumption 1, and let u_0 be any nonnegative initial datum such that $u_0 \in L^1_{\mathbb{G}^s_M}(M)$. Then there exists a weak dual solution to problem (1)*, in the sense of Definition 1.*

WDS are obtained as monotone limits of mild solutions in $L^1(M) \cap L^\infty(M)$ associated to a monotone sequence of initial data.

Main results

Let us now state the results mentioned above.

Theorem 1 (Existence of a WDS for data in $L^1_{\mathbb{G}^s_M}$)

Let M satisfy Assumption 1, and let u_0 be any nonnegative initial datum such that $u_0 \in L^1_{\mathbb{G}^s_M}(M)$. Then there exists a weak dual solution to problem (1)*, in the sense of Definition 1.*

WDS are obtained as monotone limits of mild solutions in $L^1(M) \cap L^\infty(M)$ associated to a monotone sequence of initial data.

Mild solution with "good" data enjoy well-known properties and, by adapting Bonforte-Vázquez, Nonlin. Anal. 2016, it can be shown that such solution are WDS. Fundamental properties of solutions are then proved.

Let us define the exponent $\vartheta_1 := (2s + N(m-1))^{-1}$ and state our L^1-L^∞ smoothing estimates.

Let us define the exponent $\vartheta_1 := (2s + N(m-1))^{-1}$ and state our L^1-L^∞ smoothing estimates.

Theorem 2 (Smoothing effects for data in $L^1(M)$)

Let M satisfy Assumption 1. Let u be the WDS to (1), constructed in Theorem 1, corresponding to $u_0 \in L^1(M)$, $u_0 \geq 0$. Then

$$\|u(t)\|_\infty \leq C\left(\frac{\|u(t)\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1\right) \leq C\left(\frac{\|u_0\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1\right) \quad \forall t > 0\,.$$

Let us define the exponent $\vartheta_1 := (2s + N(m-1))^{-1}$ and state our L^1-L^∞ smoothing estimates.

Theorem 2 (Smoothing effects for data in $L^1(M)$)

Let M satisfy Assumption 1. Let u be the WDS to (1), *constructed in Theorem 1, corresponding to $u_0 \in L^1(M)$, $u_0 \geq 0$. Then*

$$\|u(t)\|_\infty \leq C \left(\frac{\|u(t)\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1 \right) \leq C \left(\frac{\|u_0\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_1 \right) \quad \forall t > 0 \,.$$

If, in addition, M satisfies Assumption 2, then we have

$$\|u(t)\|_\infty \leq C \, \frac{\|u(t)\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \leq C \, \frac{\|u_0\|_1^{2s\vartheta_1}}{t^{N\vartheta_1}} \qquad \forall t > 0 \,.$$

If we require Assumption 3 as well, the bounds for long time improve:

If we require Assumption 3 as well, the bounds for long time improve:

Theorem 3

Assume that M also satisfies Assumption 3 (and $u_0 \not\equiv 0$). Then:

$$\|u(t)\|_\infty \le \frac{C}{t^{\frac{1}{m-1}}} \left[\log\left(t\,\|u_0\|_1^{m-1}\right)\right]^{\frac{s}{m-1}} \qquad \forall t \ge t_0(u_0)$$

If we require Assumption 3 as well, the bounds for long time improve:

Theorem 3

Assume that M also satisfies Assumption 3 (and $u_0 \not\equiv 0$). Then:

$$\|u(t)\|_\infty \le \frac{C}{t^{\frac{1}{m-1}}} \left[\log\left(t\,\|u_0\|_1^{m-1}\right)\right]^{\frac{s}{m-1}} \qquad \forall t \ge t_0(u_0)$$

In fact, the long-time behaviour even of the linear, non-fractional heat equation is faster than in $\mathbb{R}^N$ (exponential!).

If we require Assumption 3 as well, the bounds for long time improve:

Theorem 3

Assume that M also satisfies Assumption 3 (and $u_0 \not\equiv 0$). Then:

$$\|u(t)\|_\infty \le \frac{C}{t^{\frac{1}{m-1}}} \left[\log\left(t\,\|u_0\|_1^{m-1}\right)\right]^{\frac{s}{m-1}} \qquad \forall t \ge t_0(u_0)$$

In fact, the long-time behaviour even of the linear, non-fractional heat equation is faster than in $\mathbb{R}^N$ (exponential!).

A similar behaviour has been noticed on $\mathbb{H}^N$ and related manifolds, in the non-fractional, non-linear situation, by Vázquez, JMPA 2015, G., Muratori, Vázquez, Adv. Math. 2017, G., Muratori, Vázquez, Math. Ann. 2019. The corresponding bounds are sharp when $s = 1$. We don't know if they are here (no known Barenblatt, nor barriers!).

When enlarging the class of allowed initial data, i.e. when dealing with the space $L^1_{\mathbb{G}^s_M}(M)$ in place of $L^1(M)$, we obtain the following $L^1_{\mathbb{G}^s_M}$-L^∞ smoothing estimates.

When enlarging the class of allowed initial data, i.e. when dealing with the space $L^1_{\mathbb{G}^s_M}(M)$ in place of $L^1(M)$, we obtain the following $L^1_{\mathbb{G}^s_M}$-L^∞ smoothing estimates.

Theorem 4 (Smoothing effects for data in $L^1_{\mathbb{G}^s_M}$)

Let M satisfy Assumption 1. Let u be the WDS to (1), constructed in Theorem 1, corresponding to $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, $u_0 \geq 0$. Then:

$$\|u(t)\|_{L^\infty(M)} \leq C_1 \left(\frac{\|u(t)\|_{L^1_{\mathbb{G}^s_M}}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right) \leq C_2 \left(\frac{\|u_0\|_{L^1_{\mathbb{G}^s_M}}^{2s\vartheta_1}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right)$$

When enlarging the class of allowed initial data, i.e. when dealing with the space $L^1_{\mathbb{G}^s_M}(M)$ in place of $L^1(M)$, we obtain the following $L^1_{\mathbb{G}^s_M}$-L^∞ smoothing estimates.

Theorem 4 (Smoothing effects for data in $L^1_{\mathbb{G}^s_M}$)

Let M satisfy Assumption 1. Let u be the WDS to (1), constructed in Theorem 1, corresponding to $u_0 \in L^1_{\mathbb{G}^s_M}(M)$, $u_0 \geq 0$. Then:

$$\|u(t)\|_{L^\infty(M)} \leq C_1 \left(\frac{\|u(t)\|^{2s\vartheta_1}_{L^1_{\mathbb{G}^s_M}}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right) \leq C_2 \left(\frac{\|u_0\|^{2s\vartheta_1}_{L^1_{\mathbb{G}^s_M}}}{t^{N\vartheta_1}} \vee \|u_0\|_{L^1_{\mathbb{G}^s_M}} \right)$$

If M also satisfies Assumption 2 (and $u_0 \not\equiv 0$), then

$$\|u(t)\|_{L^\infty(M)} \leq C_3 \, \frac{\|u_0\|^{\frac{1}{m}}_{L^1_{\mathbb{G}^s_M}}}{t^{\frac{1}{m}}} \qquad \forall t \geq \|u_0\|^{-(m-1)}_{L^1_{\mathbb{G}^s_M}} .$$

Remark

- *It is remarkable that the exponents in* (4) *are the Euclidean ones corresponding to the unweighted* L^1 *space, though even in* $\mathbb{R}^N$ *the space of data is larger.*

Remark

- *It is remarkable that the exponents in* (4) *are the Euclidean ones corresponding to the unweighted* L^1 *space, though even in* $\mathbb{R}^N$ *the space of data is larger.*
- *It is however an open problem to determine the largest possible class of data for which solutions, possibly in the distributional sense, exist.*

Remark

- *It is remarkable that the exponents in* (4) *are the Euclidean ones corresponding to the unweighted* L^1 *space, though even in* $\mathbb{R}^N$ *the space of data is larger.*
- *It is however an open problem to determine the largest possible class of data for which solutions, possibly in the distributional sense, exist.*

  This problem has been solved in the Euclidean, non-fractional case in Bénilan, Crandall, Pierre, Indiana 1984, and "almost solved" in certain class of manifolds in G., Muratori, Punzo, JMPA 2018. The fractional setting is still not completely solved even in the Euclidean case.

Remark

- *It is remarkable that the exponents in* (4) *are the Euclidean ones corresponding to the unweighted* L^1 *space, though even in* $\mathbb{R}^N$ *the space of data is larger.*
- *It is however an open problem to determine the largest possible class of data for which solutions, possibly in the distributional sense, exist.*

  This problem has been solved in the Euclidean, non-fractional case in Bénilan, Crandall, Pierre, Indiana 1984, and "almost solved" in certain class of manifolds in G., Muratori, Punzo, JMPA 2018. The fractional setting is still not completely solved even in the Euclidean case.

It is impossible to enter into details of the proof. But it might be instructive to state a couple of crucial Lemmata, to have a hint of the necessary tools.

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

Lemma

Let M satisfy Assumption 3 for some $\kappa \geq 0$, and let M_κ be the space form of curvature equal to $-\kappa$, m_{M_κ} its volume measure and $\mathbb{G}^s_{M_\kappa}$ its fractional Green function.

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

Lemma

Let M satisfy Assumption 3 for some $\kappa \geq 0$, and let M_κ be the space form of curvature equal to $-\kappa$, m_{M_κ} its volume measure and $\mathbb{G}^s_{M_\kappa}$ its fractional Green function. Then, for all $r > 0$ and all $o \in M$, we have

$$\int_{B_r(o)} \mathbb{G}^s_M(x, o)\, dm(x) \leq \int_{B_r(o_c)} \mathbb{G}^s_{M_\kappa}(x, o_c)\, dm_{M_\kappa}(x)\,,$$

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

Lemma

Let M satisfy Assumption 3 for some $\kappa \geq 0$, and let M_κ be the space form of curvature equal to $-\kappa$, m_{M_κ} its volume measure and $\mathbb{G}^s_{M_\kappa}$ its fractional Green function. Then, for all $r > 0$ and all $o \in M$, we have

$$\int_{B_r(o)} \mathbb{G}^s_M(x, o)\, dm(x) \leq \int_{B_r(o_c)} \mathbb{G}^s_{M_\kappa}(x, o_c)\, dm_{M_\kappa}(x)\,,$$

where o_κ stands for any pole in M_κ and $B_r(o_\kappa) \subset M_\kappa$ for the geodesic ball of radius r in centered at o_κ.

Assume Assumptions 2 or 3. It is first fundamental to compare the Green function on M (and its integrals over geodesic balls) with the Green function on the associated space form M_κ (i.e. the hyperbolic space of constant curvature $-\kappa$, or $\mathbb{R}^n$ if $\kappa = 0$).

Lemma

Let M satisfy Assumption 3 for some $\kappa \geq 0$, and let M_κ be the space form of curvature equal to $-\kappa$, m_{M_κ} its volume measure and $\mathbb{G}^s_{M_\kappa}$ its fractional Green function. Then, for all $r > 0$ and all $o \in M$, we have

$$\int_{B_r(o)} \mathbb{G}^s_M(x, o)\, dm(x) \leq \int_{B_r(o_c)} \mathbb{G}^s_{M_\kappa}(x, o_c)\, dm_{M_\kappa}(x)\,,$$

where o_κ stands for any pole in M_κ and $B_r(o_\kappa) \subset M_\kappa$ for the geodesic ball of radius r in centered at o_κ. Furthermore, we also have that

$$\mathbb{G}^s_M(x, y) \leq \mathbb{G}^s_{M_\kappa}(x_\kappa, y_\kappa)$$

for all $x, y \in M$ and their corresponding transplanted points $x_\kappa, y_\kappa \in M_c$ with respect to polar coordinates centered at o and o_κ, respectively.

The above Lemma is nontrivial since when requiring a curvature bound $\mathbb{G}^s_M$ and the volume measure have opposite monotonicity.

The above Lemma is nontrivial since when requiring a curvature bound $\mathbb{G}_M^s$ and the volume measure have opposite monotonicity. To solve the issue, it is necessary to use the representation of the fractional Green function in terms of the semigroup:

$$\mathbb{G}_M^s(x,y) := c\int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

The above Lemma is nontrivial since when requiring a curvature bound $\mathbb{G}_M^s$ and the volume measure have opposite monotonicity. To solve the issue, it is necessary to use the representation of the fractional Green function in terms of the semigroup:

$$\mathbb{G}_M^s(x,y) := c\int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

so that

$$\int_{B_r(o)} \mathbb{G}_M^s(y,o)\,\mathrm{d}m(y) = c\int_0^{+\infty} \frac{1}{t^{1-s}}\left(\int_{B_r(o)} k_M(t,y,o)\,\mathrm{d}m(y)\right)\mathrm{dt}.$$

The above Lemma is nontrivial since when requiring a curvature bound $\mathbb{G}_M^s$ and the volume measure have opposite monotonicity. To solve the issue, it is necessary to use the representation of the fractional Green function in terms of the semigroup:

$$\mathbb{G}_M^s(x,y) := c\int_0^{+\infty} \frac{k_M(t,x,y)}{t^{1-s}}\,\mathrm{dt}$$

so that

$$\int_{B_r(o)} \mathbb{G}_M^s(y,o)\,\mathrm{d}m(y) = c\int_0^{+\infty} \frac{1}{t^{1-s}}\left(\int_{B_r(o)} k_M(t,y,o)\,\mathrm{d}m(y)\right)\mathrm{dt}.$$

One then notice that $\int_{B_r(o)} k_M(t,y,o)\,\mathrm{d}m(y)$ solves

$$\begin{cases} \partial_t u = \Delta_M u & \text{in } M\times(0,+\infty)\,, \\ u(0,\cdot) = \chi_{B_r(o)} & \text{in } M\,. \end{cases}$$

and concludes using known Hessian comparison Theorems.

The next (and last!) Lemma might look obvious, but is very delicate: fractional potentials behave like the fractional Green function at infinity.

The next (and last!) Lemma might look obvious, but is very delicate: fractional potentials behave like the fractional Green function at infinity.

Lemma

Let M satisfy Assumption 1. Let $\psi \in L_c^\infty(M)$ be nonnegative and s.t. $\mathrm{supp}(\psi) \subseteq B_\sigma(x_0)$ *for some* $0 < \sigma < 1$ *and* $x_0 \in M$.

The next (and last!) Lemma might look obvious, but is very delicate: fractional potentials behave like the fractional Green function at infinity.

Lemma

Let M satisfy Assumption 1. Let $\psi \in L_c^\infty(M)$ be nonnegative and s.t. $\mathrm{supp}(\psi) \subseteq B_\sigma(x_0)$ for some $0 < \sigma < 1$ and $x_0 \in M$. Then:

$$\underline{C}\,\|\psi\|_1 \left(1 \wedge r(x_0, x)^{N-2s}\right) \mathbb{G}_M^s(x, x_0) \le$$
$$\le (-\Delta_M)^{-s}\psi(x) \le \overline{C}\,\|\psi\|_\infty\, \sigma^N\, \mathbb{G}_M^s(x, x_0) \quad \forall x \in M \setminus \{x_0\}\,.$$

The next (and last!) Lemma might look obvious, but is very delicate: fractional potentials behave like the fractional Green function at infinity.

Lemma

Let M satisfy Assumption 1. Let $\psi \in L^\infty_c(M)$ *be nonnegative and s.t.* $\mathrm{supp}(\psi) \subseteq B_\sigma(x_0)$ *for some* $0 < \sigma < 1$ *and* $x_0 \in M$*. Then:*

$$\underline{C}\,\|\psi\|_1 \left(1 \wedge r(x_0,x)^{N-2s}\right) \mathbb{G}^s_M(x,x_0) \leq$$
$$\leq (-\Delta_M)^{-s}\psi(x) \leq \overline{C}\,\|\psi\|_\infty\,\sigma^N\,\mathbb{G}^s_M(x,x_0) \quad \forall x \in M \setminus \{x_0\}\,.$$

The dependence on the radius σ is needed. The proof depends strongly on Li-Yau estimates: if v is a positive solution to the heat equation on M, then

$$v(t_1,x_1) \leq c_0 \left(\frac{t_2}{t_1}\right)^\beta v(t_2,x_2)\,e^{c_1\frac{r(x_1,x_2)}{t_2-t_1}+c_2(t_2-t_1)}$$

for all $0 < t_1 < t_2 < 3$ and all $x_1, x_2 \in M$.

Open problems

- Solutions that may change sign: Extend our results to signed solutions. Also investigate whether extension methods as in Banica, González, Sáez, Rev. Mat. Iberoam. 2015, can be applied.

Open problems

- Solutions that may change sign: Extend our results to signed solutions. Also investigate whether extension methods as in Banica, González, Sáez, Rev. Mat. Iberoam. 2015, can be applied.
- Uniqueness: Show that WDS are unique, not only the ones obtained by limits of monotone approximations, as done here. Such result is known from G., Muratori, Punzo, Calc. Var. 2015 in the Euclidean case for very weak solutions.

Open problems

- Solutions that may change sign: Extend our results to signed solutions. Also investigate whether extension methods as in Banica, González, Sáez, Rev. Mat. Iberoam. 2015, can be applied.
- Uniqueness: Show that WDS are unique, not only the ones obtained by limits of monotone approximations, as done here. Such result is known from G., Muratori, Punzo, Calc. Var. 2015 in the Euclidean case for very weak solutions.
- Large data: Characterize the class of data for which a solution exists, at least on $[0, T]$.

- Large time behaviour: Prove existence of fundamental solutions, namely positive solutions taking a Dirac delta as initial datum, and investigate their role in the asymptotic behaviour of general solutions as holds in the Euclidean case (Vázquez, JEMS 2014).

- Large time behaviour: Prove existence of fundamental solutions, namely positive solutions taking a Dirac delta as initial datum, and investigate their role in the asymptotic behaviour of general solutions as holds in the Euclidean case (Vázquez, JEMS 2014).

- Large time behaviour: Prove existence of fundamental solutions, namely positive solutions taking a Dirac delta as initial datum, and investigate their role in the asymptotic behaviour of general solutions as holds in the Euclidean case (Vázquez, JEMS 2014).
- Investigate similar problems in the fast diffusion case: in progress with E. Berchio, M. Bonforte.

- Large time behaviour: Prove existence of fundamental solutions, namely positive solutions taking a Dirac delta as initial datum, and investigate their role in the asymptotic behaviour of general solutions as holds in the Euclidean case (Vázquez, JEMS 2014).
- Investigate similar problems in the fast diffusion case: in progress with E. Berchio, M. Bonforte.
- Investigate the case of *nonnegative* Ricci curvature: in progress with D, Monticelli, F, Punzo, estimates are given in terms of the volume growth of Riemannian balls.

- Large time behaviour: Prove existence of fundamental solutions, namely positive solutions taking a Dirac delta as initial datum, and investigate their role in the asymptotic behaviour of general solutions as holds in the Euclidean case (Vázquez, JEMS 2014).
- Investigate similar problems in the fast diffusion case: in progress with E. Berchio, M. Bonforte.
- Investigate the case of *nonnegative* Ricci curvature: in progress with D, Monticelli, F, Punzo, estimates are given in terms of the volume growth of Riemannian balls.

THANKS FOR YOUR ATTENTION!

Proposition

Let M satisfy Assumption 1. Let u be the WDS to (1) *corresponding to any nonnnegative initial datum $u_0 \in L^1(M) \cap L^\infty(M)$. Then, we have:*

$$\int_M u(t,x)\,\mathbb{G}^s_M(x,x_0)\,dm(x) \le \int_M u_0(x)\,\mathbb{G}^s_M(x,x_0)\;dm(x)\,,$$

for all $t \ge 0$ and all $x_0 \in M$, and

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}}(t_1-t_0)\,u^m(t_0,x_0) \le \int_M [u(t_0,x)-u(t_1,x)]\,\mathbb{G}^s_M(x,x_0)\,dm(x)$$
$$\le (m-1)\,\frac{t^{\frac{m}{m-1}}}{t_0^{\frac{1}{m-1}}}\,u^m(t,x_0)$$

for a.e. $(t_0,t_1,t,x_0) \in (\mathbb{R}^+)^3 \times M$, with $0 < t_0 \le t_1 \le t$.

Proposition

Let M satisfy Assumption 1. Let u be the WDS to (1) *corresponding to any nonnnegative initial datum $u_0 \in L^1(M) \cap L^\infty(M)$. Then, we have*

$$\|u(t)\|_{L^1_{x_0,\mathbb{G}^s_M}} \le C\,\|u_0\|_{L^1_{x_0,\mathbb{G}^s_M}} \qquad \text{for all } t \ge 0,\, x_0 \in M\,.$$

If u, v are two ordered WDS to problem (1) *corresponding to nonnegative initial data $u_0, v_0 \in L^1(M) \cap L^\infty(M)$, respectively, it holds*

$$\|u(t) - v(t)\|_{L^1_{x_0,\mathbb{G}^s_M}} \le C\,\|u_0 - v_0\|_{L^1_{x_0,\mathbb{G}^s_M}} \qquad \text{for all } t \ge 0 \text{ and all } x_0 \in M\,.$$

Furthermore, for all $0 < R \le 1$, we have, for all $t \ge 0$ and all $x_0 \in M$:

$$R^{N-2s} \int_{M \setminus B_R(x_0)} u(t,x)\, \mathbb{G}^s_M(x,x_0)\, d(x) \le C\,\|u(t)\|_{L^1_{x_0,\mathbb{G}^s_M}}\,.$$