Localization of peaks for high-order equations

Frédéric Robert
Institut Elie Cartan, Université de Lorraine (Nancy)

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Let (M, g) be a compact Riemmanian manifold of dimension $n \geq 2$, and take $k \in \mathbb{N}$ such that $n > 2k \geq 2$. We are interested in functions $u \in C^{2k}(M)$ that are solutions to

$$Pu = |u|^{2^\star - 2} u \text{ in } M \text{ with } 2^\star := \frac{2n}{n-2k}$$

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Here, $\Delta_g = -\mathrm{div}_g \nabla$. Such a PDE arises in conformal geometry:

- $\mathbf{k = 1}$, the scalar curvature equation is

$$\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \frac{n-2}{4(n-1)} R_{\tilde{g}} u^{\frac{n+2}{n-2}} \, , \; u > 0$$

  where R_g (resp. $R_{\tilde{g}}$) is the scalar curvature of g (resp. $\tilde{g} = u^{\frac{4}{n-2}} g$).
- $\mathbf{k = 2}$, the Paneitz operator connects Branson's $Q-$curvatures in a conformal class too:

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- More generally, for any $\mathbf{k \geq 1}$, there is the conformal GJMS operator P_g and a notion of $Q-$curvature

These operators are **conformally invariant** in the following sense: if $\tilde{g} = u^{\frac{4}{n-2k}} g$, then

$$P_{\tilde{g}} \varphi = u^{-(2^\star - 1)} P_g(u\varphi) \text{ for all } \varphi \in C^\infty(M)$$

The invariance/instability of the equation

When $(\mathbf{M}, \mathbf{g}) = (\mathbb{R}^{\mathbf{n}}, \xi)$ (which is not compact...) the model is

$$\Delta_{eucl}^k U = U^{2^\star - 1}, \; U > 0 \text{ in } \mathbb{R}^n.$$

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This invariance generates an **intrinsic dynamic** of the equation.

For instance, you can take

$$U(x):=\alpha_{n,k}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-2k}{2}}\qquad U^{\frac{4}{n-2k}}=\alpha'_{n,k}\left(\frac{1}{1+|x|^2}\right)^2 \Rightarrow \text{round sphere}$$

so that

$$U_{\mu,x_0}(x):=\alpha_{n,k}\left(\frac{\mu}{\mu^2+|x-x_0|^2}\right)^{\frac{n-2k}{2}}$$

so that

$$\lim_{\mu\to0}U_{\mu,x_0}(x_0)=+\infty \text{ and } \lim_{\mu\to0}U_{\mu,x_0}(x)=0 \text{ for all } x\neq x_0$$

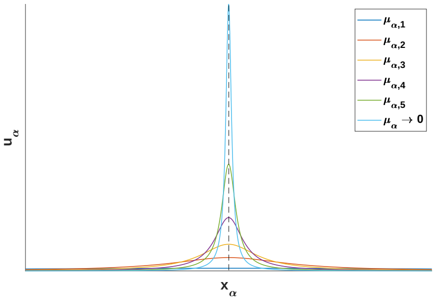

Figure: $\lim_{\mu\to 0} U_{\mu,x_0}(x_0) = +\infty$ and $\lim_{\mu\to 0} U_{\mu,x_0}(x) = 0$ for all $x \neq x_0$

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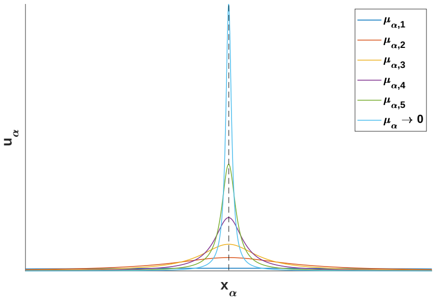
$\Rightarrow$ **Instability**.



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And they are going to be our model to describe instability
But there can be other types of peaks.

Definition (Exponential chart)

A smooth exponential chart $\tilde{\exp}$ around $p_0 \in M$ is a function

$$\begin{array}{rccc} \tilde{\exp}_p : & \mathbb{R}^n & \to & M \\ & (X^1, ..., X^n) & \mapsto & \exp_p(\sum_i X^i E_i(p)) \end{array}$$

where $\exp_p : T_pM \to M$ is the usual exponential map and $(E_i(p))_{i=1,\ldots,n}$ is a smooth orthonormal basis of T_pM, p close to p_0.

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We say that a family $B = (B_\alpha)_\alpha \in H_k^2(M)$ is a Peak centered at $(x_\alpha)_\alpha \in M$ with radius $(\mu_\alpha)_\alpha \to 0$ if there exists $U \in D_k^2(\mathbb{R}^n)$, $U \not\equiv 0$, and an exponential chart $\tilde{\exp}$ around $x_0 := \lim_{\alpha \to 0} x_\alpha$ and a cutoff function (η_α) such that

$$B_\alpha(x) = \eta(x)\mu_\alpha^{-\frac{n-2k}{2}} U\left(\frac{\tilde{\exp}_{x_\alpha}^{-1}(x)}{\mu_\alpha}\right) + o(1) \text{ in } H_k^2(M). \quad (1)$$

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There are **examples** of solutions to our PDE blowing-up like peaks (Pistoia et al, Vétois et al, Casteras-Bakri). We prove here that **any blowing-up solutions behave like a peak** and has a precise **localization**:

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Theorem (R., 2023)

Consider a family $(u_\alpha)_\alpha \in C^{2k}(M)$ *such that*

$$\Delta_g^k u_\alpha + (-1)^{k-1}\nabla^{k-1}(A_\alpha \nabla^{k-1} u_\alpha) + lot = |u_\alpha|^{2^\star-2} u_\alpha \text{ in } M, \text{ for all } \alpha > 0.$$

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The main result

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Moreover,

$$|u_\alpha(x)| \le C\left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x, x_\alpha)^2}\right)^{\frac{n-2k}{2}}$$

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$\Rightarrow$ $\mathrm{Weyl}_g \otimes B$ arises only when dealing with sign-changing u_α in the non-lcf setting

The case of small dimension $n = 2k + 1$

If $u_\alpha = B_\alpha + o(1)$ for a bubble $B = (B_\alpha)_\alpha$ where $P_\alpha u_\alpha = |u_\alpha|^{2^\star - 2} u_\alpha$, then

$$\left(\int_{\mathbb{R}^n} |U|^{2^\star - 2} U \, dX\right) m_{P_\infty}(x_0) = 0.$$

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This is possible only for some sign-changing U, but not all of them.

Theorem (R., 2023)

$$\left\{ \begin{array}{c} P_\alpha u_\alpha = |u_\alpha|^{2^\star - 2} u_\alpha \ \textit{in } M \\ + \textit{ one peak Blow-up} \end{array} \right\} \Rightarrow$$

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Result+comments (localization)

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The key is to get the pointwise control.

Result+comments (pointwise control)

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- **Here:** we want a method in analysis that does not require geometric assumptions or sign assumptions... because it simply more natural.
- The main difficulty: **how to bypass the maximum principle?**

Proof. Step 1: rescaling

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Objective: We want $(\star)$ on all the manifold M

Proof. Step 2: the final argument in 1 page!

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$$u_\alpha(x) \leq C \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x, x_\alpha)^2} \right)^{\frac{n-2k}{2}} \text{ in } M - B(x_\alpha, 2R\mu_\alpha) : \text{ DONE!}$$

Proof. Step 2: the final argument in 1 page!

- Write $P_\alpha u_\alpha = u_\alpha^{2^\star-1}$ as a **linear problem**

$$P_\alpha u_\alpha = V_\alpha u_\alpha \text{ with } V_\alpha = u_\alpha^{2^\star-2}$$

- Let G_α be the **Green's function** for $P_\alpha - V_\alpha$
- For $x \in M$ s.t $d(x, x_\alpha) > 2R\mu_\alpha$, write **Green's representation** on $M - B(x_\alpha, R\mu_\alpha)$:

$$u_\alpha(x) = \int_{\partial B(x_\alpha, R\mu_\alpha)} \sum_{i<2k} \nabla^i G_\alpha(x,\cdot) \star \nabla^{2k-1-i} u_\alpha \, dv_g$$

- We know u_α on the boundary: $u_\alpha(z) \simeq \mu_\alpha^{-\frac{n-2k}{2}}$, similar for derivatives.
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Except that this does not work...

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Intermission: $V_\alpha = u_\alpha^{2^\star-2}$ is a Hardy-potential

Let us go back to the invariance. A paramount quantity here is

$$w_\alpha(x) := d_g(x,x_\alpha)^{\frac{n-2k}{2}} u_\alpha(x) \qquad (\star)$$

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So $\forall \epsilon > 0$, $\exists R_\epsilon > 0$ such that $|V_\alpha(x)| \le \frac{\epsilon}{d_g(x,x_\alpha)^{2k}}$ for all $d_g(x,x_\alpha) > R_\epsilon \mu_\alpha$

Our equation rewrites

$$(P_\alpha - V_\alpha)u_\alpha = 0$$

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- When $x, y \in M$ are far from the singularity x_α, then

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Back to the proof: What we consider now

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- For general x, y: a mix of these two cases.

We get a **sharp control** of the Green's function and of its derivatives

At the end of the day, we have proved that

Theorem (R., 2022)

$$\left.\begin{cases} P_\alpha u_\alpha = |u_\alpha|^{2^\star - 2} u_\alpha \ \textit{in } M \\ + \textit{Blow-up} \\ + \textit{Minimal energy} \end{cases}\right\} \Rightarrow |u_\alpha(x)| \le C \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x, x_\alpha)^2} \right)^{\frac{n-2k}{2}}$$

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Proof of the localization via Pohozaev's identity

For any function $v : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, we have that

$$\int_\Omega \left(x^i \partial_i v + \frac{n-2k}{2} v \right) \left(\Delta_\xi^k v - |v|^{2^\star - 2} v \right) \, dx = \int_{\partial\Omega} \dots$$

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In the Pohozaev identity, we then get

$$\int_\Omega \left(x^i \partial_i u_\alpha + \frac{n-2k}{2} u_\alpha \right) \left((\Delta_\xi^k - \Delta_{\tilde{g}}^k) u_\alpha - (-1)^{k-1} \nabla^{k-1} ((A_\alpha - A_{GJMS}) \nabla^{k-1} u_\alpha) \right) dx = \ldots$$

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In the Pohozaev identity, we then get

$$\int_\Omega T(u_\alpha)\left(\underbrace{(\Delta_\xi^k - \Delta_{\tilde{g}}^k)u_\alpha}_{\text{measures } \tilde{g}-\xi} - (-1)^{k-1}\nabla^{k-1}(\underbrace{(A_\alpha - A_{GJMS})}_{\text{distance from the conf.op.}}\nabla^{k-1}u_\alpha)\right)dx = \ldots$$

where $T(u_\alpha) := x^i\partial_i u_\alpha + \frac{n-2k}{2}u_\alpha$.

For any function $v : \mathbb{R}^n \to \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$, we have that

$$\int_\Omega \left(x^i \partial_i v + \frac{n-2k}{2} v\right) \left(\Delta_\xi^k v - |v|^{2^\star - 2} v\right) dx = \int_{\partial\Omega} \cdots$$

Our equation is

$$\Delta_g^k u_\alpha + (-1)^{k-1} \nabla^{k-1}(A_\alpha \nabla^{k-1} u_\alpha) + lot = |u_\alpha|^{2^\star - 2} u_\alpha \text{ in } M$$

we write it as

$$P_g u_\alpha + (-1)^{k-1} \nabla^{k-1}((A_\alpha - A_{GJMS}) \nabla^{k-1} u_\alpha) + lot = |u_\alpha|^{2^\star - 2} u_\alpha \text{ in } M$$

we can change the metric in metric $\tilde{g}$ conformal to g becomes "almost flat", that is $Ric_{\tilde{g}}(x_\alpha) = 0$. We then write

$$\Delta_{\tilde{g}}^k u_\alpha + (-1)^{k-1} \nabla^{k-1}((A_\alpha - A_{GJMS}) \nabla^{k-1} u_\alpha) + lot = |u_\alpha|^{2^\star - 2} u_\alpha \text{ in } M$$

In the Pohozaev identity, we then get

$$\int_\Omega T(u_\alpha) \left(\underbrace{(\Delta_\xi^k - \Delta_{\tilde{g}}^k) u_\alpha}_{\text{measures } \tilde{g} - \xi} - (-1)^{k-1} \nabla^{k-1} \Big(\underbrace{(A_\alpha - A_{GJMS})}_{\text{distance from the conf.op.}} \nabla^{k-1} u_\alpha \Big) \right) dx = \ldots$$

where $T(u_\alpha) := x^i \partial_i u_\alpha + \frac{n-2k}{2} u_\alpha$. When $n > 2k+2$, we get

$$\mathrm{Weyl}_g \otimes B + \int_{\mathbb{R}^n} (A_\infty - A_{GJMS})_{x_0} \left(\nabla^{k-1} U, \nabla^{k-1} U\right) dX = 0$$

On (M, g) of dimension $n \geq 5$, see Hebey, there exists $B > 0$ such that the following Sobolev inequality holds:

$$\left(\int_M |u|^{\frac{2n}{n-4}} \, dv_g \right)^{\frac{n-4}{n}} \leq K_4(n) \left(\int_M (\Delta_g u)^2 \, dv_g + B \|u\|^2_{H_1^2} \right) \text{ for all } u \in H_2^2(M). \quad (I_B)$$

where $K_4(n)$ is the optimal Euclidean constant.

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where $K_4(n)$ is the optimal Euclidean constant.Let $B_0(g)$ be the smallest number B such that this inequality holds for all $u \in H_2^2(M)$.

Theorem

Assume that $n \geq 6$. Then if there is no nontrivial extremal for $(I_{B_0(g)})$, then

$$B_0(g) = \frac{3n^2 - 6n - 12}{6n(n-1)} max_{x \in M} R_g(x).$$

$$Pa_g u = \Delta_g^2 u - div_g \left[(a_n S_g g + b_n Ric_g)^{\#} du \right] + \frac{n-4}{2} Q_g u,$$

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \quad , \quad b_n = -\frac{4}{n-2} \; ,$$

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g R_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2 - \frac{2}{(n-2)^2} |Ric_g|_g^2 .$$