The L^p-positivity preservation on Riemannian manifolds

Giona Veronelli

Università di Milano-Bicocca

Grenoble

Sobolev Inequalities in the Alps

June 30th, 2023

The property we are interested in this talk:

Definition (positivity preservation)

A Riemannian manifold (M, g) is said to be L^p-Positivity Preserving (L^p-PP in short) if, for any $\lambda > 0$,

$$(L^p\text{-PP}) \qquad \begin{cases} -\Delta w + \lambda w \geq 0 \text{ distributionally on } M \\ w \in L^p(M) \end{cases} \quad \Rightarrow \quad w \geq 0.$$

- i.e., $(-\Delta + \lambda)^{-1} : \cdots \to L^p$ is positive
- For the moment think (M, g) smooth and complete. In the second part we will try to remove these assumptions
- Here $\Delta = \text{div}(\nabla \cdot)$ is the negative Laplace-Beltrami operator (i.e. $\Delta = +\frac{d^2}{dx^2}$ on $\mathbf{R}$)
- $-\Delta w + \lambda w \geq 0$ distributionally means

$$\int_M w(-\Delta + \lambda)\varphi \geq 0, \qquad \forall\, 0 \leq \varphi \in C_c^\infty(M)$$

In particular $-\Delta w + \lambda w = \mu$ is a positive Radon measure.

In this talk:

- Motivations for considering the L^p-PP property
- Older approaches and techniques
- The case of complete manifolds and the BMS conjecture
- Removing compact sets from complete manifolds
- Dirichlet metric measure spaces (and a new notion of subharmonicity)

Based on

- [S. Pigola and G. Veronelli, *L^p Positivity Preserving and a conjecture by M. Braverman, O. Milatovic and M. Shubin*, 2021]

unpublished, and incorporated in

- [S. Pigola, D. Valtorta and G. Veronelli, *Approximation, regularity and positivity preservation on Riemannian manifolds*. arXiv:2301.05159, submitted for publication
- [B. Güneysu, S. Pigola, P. Stollmann, and G. Veronelli, *A new notion of subharmonicity on locally smoothing spaces, and a conjecture by Braverman, Milatovic, Shubin.* arXiv:2302.09423, submitted for publication]

L^p-**Positivity preservation:** name introduced by B. Güneysu, but the property goes back to the celebrated work of T. Kato on the spectral theory of Schrödinger operators with singular potential.

A step back:

Theorem

Let (M, g) be a complete Riemannian manifold. Then, the Laplace operator $(\Delta, C_c^\infty(M))$ is essentially self-adjoint in $L^2(M)$.

Rmk. Celebrated result which goes back to [P.M. Gaffney, Proc. Nat. Acad. Sci. (1951)] and [W. Roelcke, Math. N. (1960)]. Popularized by [R. Stricharts, JFA 1983].

Recall that $(\Delta, C_c^\infty(M))$ is **essentially self-adjoint** in $L^2(M)$ when $\bar{\Delta} = \Delta^*$.
Start with

$$\Delta : C_c^\infty(M) \subseteq L^2(M) \to L^2(M)$$

Δ is densely defined and symmetric.

Its **adjoint**: $\Delta^* : \mathcal{D}(\Delta^*) \subseteq L^2(M) \to L^2(M)$
is defined on the domain

$$\mathcal{D}(\Delta^*) = \{u \in L^2(M) : \Delta u \in L^2(M)\}$$

Its **closure**: $\bar{\Delta} : \mathcal{D}(\bar{\Delta}) \subseteq L^2(M) \to L^2(M)$
(defined by the equation $\overline{\Gamma(\Delta)} = \Gamma(\bar{\Delta})$ on graphs) has domain

$$\mathcal{D}(\bar{\Delta}) = \overline{C_c^\infty(M)}^{\|\cdot\|_{\widetilde{W}^{2,2}}}$$

where

$$\|u\|^2_{\widetilde{W}^{2,2}} = \|u\|^2_{L^2} + \|\Delta u\|^2_{L^2}.$$

We have

$$(\Delta, C_c^\infty(M)) \subseteq (\bar{\Delta}, \mathcal{D}(\bar{\Delta})) \subseteq (\Delta^*, \mathcal{D}(\Delta^*)).$$

Rmk. $\bar{\Delta} = \Delta^* \Leftrightarrow C_c^\infty(M)$ is dense in $\widetilde{W}^{2,2}(M)$. No direct proofs of this latter fact are known on general complete manifolds!

A great effort has been spent in order to extend this to more general operators.

Let $0 \leq V \in L^2_{loc}(M)$ and consider $\boxed{\mathscr{L} = \Delta - V : C^\infty_c(M) \subseteq L^2(M) \to L^2(M)}$

- As above, the symmetric operator $\mathscr{L}$ has a closure $\bar{\mathscr{L}} \subseteq \mathscr{L}^*$ with

$$\mathcal{D}(\bar{\mathscr{L}}) = \overline{C^\infty_c(M)}^{\widetilde{W}^{1,2}_V}, \quad \|\varphi\|^2_{\widetilde{W}^{1,2}_V} = \int_M \varphi^2 + \int_M (\Delta\varphi - V\varphi)^2.$$

  and

$$\mathcal{D}(\mathscr{L}^*) = \{u \in L^2(M) : \Delta u - Vu \in L^2(M)\}$$

- $\mathscr{L}$ is essentially self-adjoint if $\bar{\mathscr{L}} = \mathscr{L}^*$. This happens iff

$$\forall\, \lambda > 0, \qquad \begin{cases} \mathscr{L} u = \lambda u \text{ distributionally on } M \\ u \in L^2(M) \end{cases} \qquad \Rightarrow \quad u \equiv 0.$$

How to prove this latter property?

The approach by [T. Kato, Israel Math. J. 1972] in $\mathbf{R}^n$ uses a short-cut.

Suppose $\mathscr{L} u = \Delta u - Vu = \lambda u$ for some $u \in L^2(M)$

$\Longrightarrow \quad \Delta u = (V + \lambda)u$ with $u \in L^1_{loc}$ & $(V + \lambda)u \in L^1_{loc}$

$\Longrightarrow \quad \Delta|u| \geq (V + \lambda)|u| \geq 0$ (by Kato's inequality)

$\Longrightarrow \quad (-\Delta w + \lambda w) \geq -Vw \geq 0$, where $0 \geq w = -|u| \in L^2(M)$.

If M is L^2-PP, then

$$\begin{cases} (-\Delta + \lambda)w \geq 0 \text{ distributionally on } M \\ w \in L^2(M) \end{cases} \quad \Rightarrow \quad w \geq 0 \quad \Rightarrow \quad w = 0.$$

The problem is now: under which geometric assumptions M is L^2-PP ?

At some point of the history...

Using a refined integration by parts techniques, Braverman, Milatovic and Shubin [M. Braverman, O. Milatovich and M. Shubin, Uspekhi Mat. Nauk 2002, Russian Math. Surveys 2002] realized that

Theorem

On any complete Riemannian manifold, for any $0 \leq V \in L^2_{loc}$*,* $\mathscr{L} = \Delta - V : C_c^\infty(M) \subseteq L^2(M) \to L^2(M)$ *is essentially self-adjoint.*

This led them to

Conjecture (BMS)

Any smooth complete Riemannian manifold is L^2-positivity preserving.

- see also the dedicated survey [B. Güneysu, Ulmer Seminare 2016-2017, available at https://arxiv.org/pdf/1709.07463.pdf]

An L^p-version of the essential self-adjointness.

Consider the Schrödinger operator $\mathscr{L} = \Delta - V(x)$ with $0 \le V(x) \in L^p_{loc}(M)$.

One would like to know if $\mathscr{L}_{p,min} = \mathscr{L}_{p,max}$ where $\mathscr{L}_{p,min} = \overline{\mathscr{L}|_{C_c^\infty(M)}}$ and $\mathcal{D}(\mathscr{L}_{p,max}) = \{u \in L^p(M) : Vu \in L^1_{loc}(M) \text{ and } \mathscr{L}u \in L^p(M)\}$.

In this case we say that C_c^∞ is an operator core for $\mathscr{L}$ in $L^p(M)$

So far only known only under (Ricci) curvature (lower) bounds.

$\Longrightarrow$ It makes sense to investigate the L^p-positivity preservation also for $p \neq 2$.

Proving the L^p-positivity preservation

In the literature, we found two approaches to prove that $\mathbf{R}^n$ is L^p-PP.

T. Kato original argument. Let $\mathscr{S}'(\mathbf{R}^n)$ be the space of tempered distributions.

- If $u \in L^p$, then $0 \leq (-\Delta + \lambda)u \in \mathscr{S}'(\mathcal{R}^n)$.
- it is enough to prove that $(-\Delta + \lambda)^{-1}$ is positive, i.e.

$$T \in \mathscr{S}'(\mathbf{R}^n),\ T \geq 0 \quad \Longrightarrow \quad (-\Delta + \lambda)^{-1}(T) \geq 0.$$

- this is true because, when acting on $\mathscr{S}(\mathbf{R}^n)$, the operator $(-\Delta + \lambda)^{-1}$ has a positive integral kernel $g_\lambda(x) = \lambda^{n-2} g_1(\lambda x)$ where

$$g_1(x) = C|x|^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(x)$$

  and K is the modified Bessel function of the second kind.

Rmk. This approach looks very much sensitive of the structure of the Euclidean space.

E. B. Davies argument, reported in Appendix B of [M. Braverman, O. Milatovich and M. Shubin, Uspekhi Mat. Nauk 2002, Russian Math. Surveys 2002].

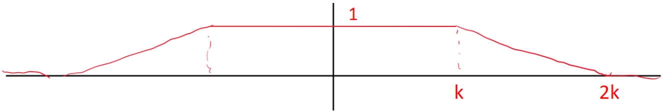
Def. Let (M, g) be a Riemannian manifold.
Say that $\{\varphi_k\} \subseteq C^2_c(M)$ is a sequence of Laplacian cut-offs if

(a) $\varphi_k \nearrow 1$, (b) $\|\nabla\varphi_k\|_\infty \to 0$ and (c) $\sup_k \|\Delta\varphi_k\|_\infty < +\infty$.

Rmk.

- For $M = \mathbf{R}^n$, take

$$\varphi_k = 1 \text{ on } B_k(0), \qquad \varphi_k = 0 \text{ on } \mathbf{R}^n \setminus B_{2k}(0) \qquad \text{and} \qquad \partial_r\varphi_k \asymp k^{-1}.$$



- On a complete manifold (M, g) the existence of Laplacian cut-offs is related to *curvature*. The most general condition is due to [D. Bianchi and A.G. Setti, Calc. Var. 2018]: $\mathrm{Ric} \geq -K^2(1 + \mathrm{dist}(x, o)^2)$, for some origin $o \in M$.

Let's see how to use Laplacian cut-offs to prove L^p-PP...

- The assumption $(-\Delta+\lambda)u \geq 0$ means: $\forall \psi \in C_c^\infty(M)$ with $\psi \geq 0$,

$$(*) \qquad \int_M u(-\Delta\psi + \lambda\psi) \geq 0.$$

- We want to prove $u \geq 0$: $\forall \eta \in C_c^\infty(M)$ with $\eta \geq 0$,

$$\int_M u\eta \geq 0.$$

- The equation $(-\Delta+\lambda)v = \eta$ has 1! solution $0 \leq v \in C^\infty(M) \cap W^{1,2}(M)$.
- If $v \in C_c^\infty$ take $\psi = v$ and we are done. Otherwise...
- Let φ_k be a sequence of Laplacian cut-offs. Evaluate $(*)$ along $\psi_k = v\varphi_k$ and compute:

$$\begin{aligned} 0 &\leq \int_M u(-\Delta\psi_k + \lambda\psi_k) \\ &= \int_M u(-\Delta v + \lambda v)\varphi_k - \int_M u\, v\, \Delta\varphi_k - 2\int_M u\langle \nabla v, \nabla\varphi_k\rangle \\ &= \int_M u\,\eta\,\varphi_k - \int_M u\, v\, \Delta\varphi_k - 2\int_M u\langle \nabla v, \nabla\varphi_k\rangle \overset{k\to+\infty}{\longrightarrow} \int_M u\,\eta\,\mathrm{d}x. \end{aligned}$$

In particular a complete Riemannian manifold (M, g) is L^p-PP when

- Ricci decay to $-\infty$ at most quadratically and $1 \leq p \leq +\infty$;
  - $p = 2$: [D. Bianchi and A.G. Setti, Calc. Var. 2018]. Laplacian cut-offs give $\int_M u\, v\, \Delta\varphi_k \to 0$.
  - $p \neq 2$: [B. Güneysu, Operator Theory: Advances and Applications, book 2017], [L. Marini and G. Veronelli, Preprint 2021]. A little more effort to get $\int_M u\langle\nabla v, \nabla\varphi_k\rangle \to 0$
- (M, g) is Cartan-Hadamard, Ricci must decay to $-\infty$ polynomially and $2 \leq p < +\infty$; [L. Marini and G. Veronelli, Preprint 2021].
  - *In this case* $\|\Delta\varphi_k\|_\infty$ *exploses polynomially,*
  - *but one disposes of super-Euclidean Hardy inequalities: u is very much* L^p*-integrable*

  $\Rightarrow \quad \int_{\mathbf{R}^n} u\, v\, \Delta\varphi_k \,\mathrm{d}x \longrightarrow 0$

Rmk. In such a setting also density of C_c^∞ in $W^{2,p}$ and L^2-Calderón-Zygmund inequalities.

A new approach

Theorem (The L^p-positivity preserving property)

Let (M, g) be a complete Riemannian manifold and let $1 < p < +\infty$. If, for some $\lambda > 0$, $u \in L^p(M)$ is a distributional solution of $(-\Delta u + \lambda u) \geq 0$ then $u \geq 0$.

Note that the L^p-PP is false in general if $p = 1, +\infty$ (but true if $Ric(x) \gtrsim -d(x)^2$)

- for $p = 1$, take a complete manifold such that $\Delta u = \delta$ has a negative L^1 solution. Then
- for $p = +\infty$,

$$M \text{ is } L^p - PP \;\Leftrightarrow M \text{ is stochastically complete;}$$

see [B. Güneysu, Birkhauser 2016] and [A. Bisterzo and L. Marini, PotA 2022].

A (formal) proof. Let $1 < p < \infty$. Suppose that

$$\begin{cases} (-\Delta + \lambda) w \geq 0 \text{ distributionally on } M, \\ w \in L^p(M) \end{cases} \Rightarrow \Delta(-w) \geq \lambda(-w)$$

Assume $\Delta(-w) \in L^1_{loc}$. Kato's inequality gives $\Delta(-w)_+ \geq \lambda(-w)_+ \geq 0$, i.e. $v = (-w)_+$ is a nonnegative subharmonic function in L^p.

Assume moreover that $v, v^{p/2} \in W^{1,2}_{loc}(M)$, then [S.T. Yau, Indiana 1976] proved

Theorem (L^p-Liouville)

Assume $1 < p < \infty$ and

$$\begin{cases} \Delta v \geq 0 \\ 0 \leq v \in L^p \\ v \text{ and } v^{p/2} \in W^{1,2}_{loc} \end{cases} \Longrightarrow \quad v \equiv const.$$

Thus $w \geq 0$

Proof (of the L^p-Liouville).

Since $v \in W^{1,2}_{loc}$ then $\Delta v \geq 0$ means

$$-\int_M \langle \nabla v, \nabla \psi \rangle \geq 0, \qquad \forall 0 \leq \psi \in W^{1,2}_c$$

Choose $\psi = (v+\delta)^{p-1}\varphi_k^2$, with $0 \leq \varphi_k \in C_c^\infty$

Integrating and elaborating (via Young inequality for any $0 < \varepsilon < 1$) as $\delta \to 0$ gives:

$$\text{Caccioppoli ineq:} \qquad C(p,\varepsilon)\int_M |\nabla v^{p/2}|^2 \varphi_k^2 \leq \int_M v^p |\nabla \varphi_k|^2.$$

If M is complete, choosing $\varphi_k \nearrow 1$, $\|\nabla \varphi_k\|_\infty \to 0$ gives $v^{p/2} \equiv const.$ □

□

Problems in the formal proof:

- We assumed $\Delta(-w) \in L^1_{loc}$ to apply Kato's inequality
- We assumed $v^{s/2} \in W^{1,2}_{loc}(M)$, $s = 2, p$, to apply the L^p-Liouville theorem

In the following:

- We will give a version of the Kato's inequality under weaker regularity assumptions
- We will show that

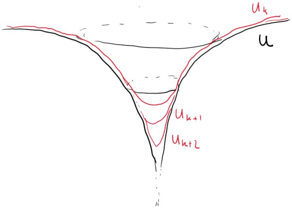
$$\begin{cases} \Delta v \geq 0 \\ v \geq 0 \end{cases} \quad \Rightarrow v^{s/2} \in W^{1,2}_{loc}(M) \text{ for any } s > 1.$$

Basic tool: the monotonic approximation of distributional subsolutions.

Theorem (monotonic approximation)

Let $\Omega \subset \Omega' \subset M$ smooth compact domains. Let $u \in L^1_{loc}(M)$ λ-subharmonic in Ω' for some $0 \leq \lambda$, i.e. $\Delta u - \lambda u \geq 0$ distributionally.
Then, there exists a sequence $\{u_k\} \subset C^\infty(\bar{\Omega})$ such that:

1. $u_k(x) \to u(x)$ *pointwise a.e. in* Ω *and in* $L^1(\Omega)$.
2. $\Delta u_k - \lambda u_k \geq 0$ *in* Ω, $\forall k \in \mathbf{N}$.
3. $u_k \geq u_{k+1} \geq u$, $\forall k \in \mathbf{N}$.



About the proof of the monotonic approximation.

WLOG we can assume $\lambda = 0$. Indeed:

- Let $0 < \alpha \in C^\infty(\Omega')$ solve $\Delta\alpha = \lambda\alpha$. Protter-Weinberger trick:

$$\Delta u - \lambda u \geq 0 \Leftrightarrow \Delta_{\tilde{g}}\left(\frac{u}{\alpha}\right) \geq 0, \qquad \text{where } \tilde{g} = \alpha^{\frac{4}{2-n}} g.$$

Case: Ω is a coordinate chart:

a) First proof by [W. Littman, ASNSP 1963] (with some gaps?)

b) Different approach via potential theory à la Hervé-Brelot [P. Sjogren, Ark. Mat. 1973], [A. Bonfiglioli and A. Lanconelli, JEMS 2013]

General (bounded) Ω:

- b) "globalized" by [A. Bisterzo and L. Marini, PotA 2022]
- we give a self-contained direct proof similar to Littman

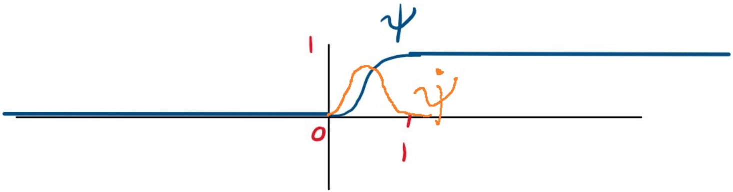
In $\mathbf{R}^n$, $u_k(x) \approx \frac{1}{|\partial B_\epsilon(x)|} \int_{\partial B_\epsilon(x)} u(y)$. On M the distance is replaced by a function of G, the Green function of Ω':

$$\Delta_x G(x,y) = -\delta_{x=y}, \qquad G(x,y) = 0 \text{ if } y \in \partial\Omega',\ x \in \mathring{\Omega}$$

Namely,

$$u_k(x) = \int_M \dot\psi(G(x,y) - k)u(y)|\nabla G(x,y)|^2$$

with



The Brezis-Kato inequality. Recall Kato inequality:

$$u \in L^1_{loc}(M) \text{ and } \Delta u \in L^1_{loc}(M) \implies \Delta u_+ = 1_{\{u>0\}}\Delta u.$$

A variant of Kato inequality for the Laplacian in $\mathbf{R}^n$ is due to [H. Brezis, Appl. Math. Optim. 1984].

Theorem (Brezis-Kato)

Let (M, g) be a Riemannian manifold. If $u \in L^1_{loc}(M)$ is a distributional solution of $\Delta u \geq f \in L^1_{loc}$, then $\Delta u_+ \geq \chi_{u>0} f$.

Idea of the proof ($f = \lambda u$). Approximate u_+ with $H_\varepsilon(u_k)$, where

- $H_\varepsilon(t) \to t_+$ as $\varepsilon \to 0$
- $\{u_k\}$ is a smooth monotonic approximation of u on Ω (replaces Euclidean convolution in Brezis-Ponce's proof).

Improved regularity of positive subsolutions. A nonnegative subharmonic distribution is much more regular than it could appear.

Theorem (regularity)

Let (M, g) be a Riemannian manifold and let $0 \le u \in L^1_{loc}(M)$ be a distributional solution of $\Delta u \ge 0$. Then $u^{s/2} \in W^{1,2}_{loc}(M)$ for every $1 < s < +\infty$.

Proof.

- Let $u_k \searrow u$ be a smooth monotonic approximation of u on a domain $\Omega \Subset M$. Thus $0 \le u_k \in C^\infty(\bar{\Omega})$ satisfy $\Delta u_k \ge 0$ in Ω.
- Let $0 \le \varphi \in C_c^\infty(M)$. *Caccioppoli ineq:*

$$C(s,\varepsilon)\int_M |\nabla u_k^{s/2}|^2\varphi^2 \le \int_M u_k^s |\nabla \varphi|^2$$

Improved regularity of positive subsolutions. A nonnegative subharmonic distribution is much more regular than it could appear.

Theorem (regularity)

Let (M, g) be a Riemannian manifold and let $0 \le u \in L^1_{loc}(M)$ be a distributional solution of $\Delta u \ge 0$. Then $u^{s/2} \in W^{1,2}_{loc}(M)$ for every $1 < s < +\infty$.

Proof.

- Let $u_k \searrow u$ be a smooth monotonic approximation of u on a domain $\Omega \Subset M$. Thus $0 \le u_k \in C^\infty(\bar{\Omega})$ satisfy $\Delta u_k \ge 0$ in Ω.
- Let $0 \le \varphi \in C^\infty_c(M)$. *Caccioppoli ineq:*

$$C(s,\varepsilon)\int_M |\nabla u_k^{s/2}|^2\varphi^2 \le \int_M u_k^s|\nabla\varphi|^2 \le \int_M u_1^s|\nabla\varphi|^2$$

$\{u_k^{s/2}\}$ is bounded in $W^{1,2}(\Omega)$, hence $u_{k_h}^{s/2} \rightharpoonup u^{s/2} \in W^{1,2}(\Omega)$.

Back to the proof of L^p-Lioville.

Let $s \in (1,p)$ and $\begin{cases} v \geq 0, \quad \Delta v \geq 0, \\ v,\ v^{s/2} \in W^{1,2}_{loc} \end{cases}$ (for free)

Caccioppoli ineq: $$C(s,\varepsilon)\int_M |\nabla v^{s/2}|^2 \varphi_k^2 \,\mathrm{d}x \leq \int_M v^s |\nabla \varphi_k|^2 \,\mathrm{d}x$$

Back to the proof of L^p-Lioville.

Let $s \in (1,p)$ and $\begin{cases} v \geq 0, \quad \Delta v \geq 0, \\ v,\ v^{s/2} \in W^{1,2}_{loc} \qquad \text{(for free)} \end{cases}$

Caccioppoli ineq:
$$C(s,\varepsilon)\int_M |\nabla v^{s/2}|^2 \varphi_k^2 \,\mathrm{d}x \leq \int_M v^s |\nabla \varphi_k|^2 \,\mathrm{d}x$$
$$\stackrel{C-S}{\leq} \left(\int_M v^p\right)^{\frac{s}{p}} \left(\int_M |\nabla \varphi_k|^{\frac{2p}{p-s}}\right)^{\frac{p-s}{p}}$$

Back to the proof of L^p-Lioville.

Let $s \in (1,p)$ and $\begin{cases} v \geq 0, \quad \Delta v \geq 0, \\ v,\ v^{s/2} \in W^{1,2}_{loc} \qquad \text{(for free)} \end{cases}$

$$\textit{Caccioppoli ineq:} \qquad C(s,\varepsilon)\int_M |\nabla v^{s/2}|^2 \varphi_k^2 \,\mathrm{d}x \leq \int_M v^s |\nabla \varphi_k|^2 \,\mathrm{d}x$$
$$\stackrel{C-S}{\leq} \left(\int_M v^p\right)^{\frac{s}{p}} \left(\int_M |\nabla \varphi_k|^{\frac{2p}{p-s}}\right)^{\frac{p-s}{p}} \stackrel{\text{as } k\to\infty}{\longrightarrow} 0$$

provided that $v \in L^p(M)$ and $\|\varphi_k\|_{\frac{2p}{p-s}} \to 0$

Such $\{\varphi_k\}_k$ exists if M is $\left(\frac{2p}{p-s}\right)$-parabolic.

Recall. q-parabolicity is a property of a space from potential theory which can be defined in term of capacities, Green functions, subharmonic functions, recurrence of the Brownian motion $(p=2)$...

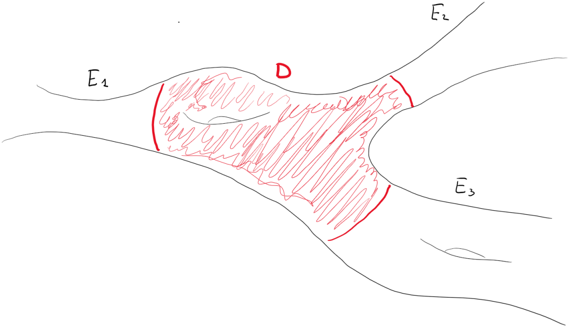
Rmk. Complete $=$ ∞-parabolic

Localizing on each end:

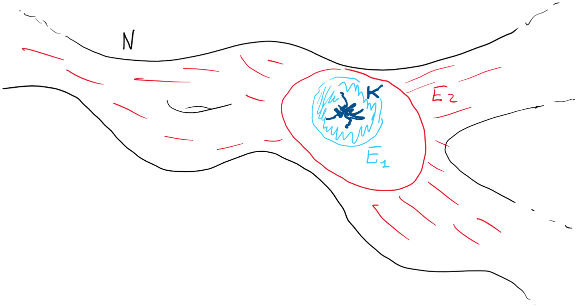
Theorem

Let $p \in (1, \infty)$. Let M be an open connected (not necessarily complete) manifold with a finite number of ends $E_1, \dots, E_N$. If each end E_j is q_j-parabolic for some $q_j \in (\frac{2p}{p-1}, \infty]$, then the L^p-Liouville holds on M, so that in particular M is L^p-PP.

Here, end = connected component of $M \setminus D$ for some compact D.



A special case. q-parabolic manifold with q large (e.g. M complete, e.g. $M = \mathbf{R}^n$) and K is a compact set.



Corollary

Let $p \in (1, \infty)$. Let N be a complete (or q-parabolic, $q > 2p/(p-1)$) Riemannian manifold and $K \subset N$ a compact set. Suppose that K is q-polar for some $q > 2p/(p-1)$. Then $M = N \setminus K$ is L^p-PP.

In potential theory: q-polar = small. It is implied by $\mathcal{H}^{n-q}(K) = 0$.

Corollary

$p \in (1,\infty)$, N complete, $\dim_{\mathcal{H}} K < n - \frac{2p}{p-1}$. Then

1. *L^p-Liouville on $N \setminus K$*
2. *$N \setminus K$ is L^p-PP*
3. *For $0 \leq V \in L^1_{loc}$,*
   $p = 2$ *$(-\Delta + V)$ is ess. self-adjoint in L^2*
   $p \neq 2$ *$C_c^\infty(N \setminus K)$ is an operator core for $(-\Delta + V)$ in L^p*

3. seems partially new even in case $V = 0$ or $p \neq 2$ and $K = \emptyset$.

Previous partial results by

- [Y. Colin de Verdière, Ann. Inst. Fourier, 1982] [J. Masamune, Comm. PDEs, 1999]: $V = 0$, $p = 2$ and K is a smooth submanifold
- [M. Hinz, J. Masamune and K. Suzuki, Nonlinear Anal. 2023]: $V = 0$ and K singular. Stronger than our result for small p and weaker for large p.
- [T. Kato, North-Holland Math. Stud, 1986],[O. Milatovic, JMAA, 2006],[B. Güneysu and S. Pigola, AMPA, 2019]: $K = \emptyset$ and geometric restriction.

What about the assumption $\dim_{\mathcal{H}} K < n - \frac{2p}{p-1}$ **?**

Proposition

If $\dim_{\mathcal{H}} K > n - \frac{2p}{p-1}$, then $N \setminus K$ does not enjoy neither the L^p-Liouville nor the L^p-PP property.

Counterexamples are modeled on solutions of $-\Delta u + \lambda u = \mu|_K$ with μ a Hausdorff measure of the suitable dimension.

Corollary. If N is complete than $N \setminus K$ is L^p-PP $\forall\, p \in (1, \infty)$ iff $K = \emptyset$.

In the threshold case:

- OK if we control also the stronger Minkowski dimension of K:

$$\mathrm{Vol}(B_\epsilon) \leq C r^{\frac{2p}{p-1}}, \qquad r \in [0, 1]$$

- In general $\dim_{\mathcal{H}} K = n - \frac{2p}{p-1}$ is not enough (counterexamples at least when $\mu_{n-\frac{2p}{p-1}}(K) = +\infty$)

Some generalizations on smooth manifolds:

- [A. Bisterzo, A. Farina and S. Pigola, arXiv:2304.00745]

$$\begin{cases}(-\Delta+\lambda)w \geq 0 \text{ distributionally on } M \\ \|w\|_{L^p(B_R)} = e^{\theta R} \text{ for } \theta = \theta(p,\lambda)\end{cases} \quad \Rightarrow \quad w \geq 0.$$

- [A. Bisterzo and G. Veronelli, work in progress] Spectral properties of $\Delta - V$ on $N \setminus K$ when $\dim_{\mathcal{H}} K > n - \frac{2p}{p-1}$ but $0 \leq V \in L^1_{loc}$ is suitably controlled

Towards metric space: yet another proof...

Let $0 \geq f \in L^1_{loc}(M)$ and let $\lambda \geq 0$. Then $\Delta f - \lambda f \geq 0$ in the sense of distributions iff $\forall\, U \Subset M$, $\forall\, t > 0$ one has

$$\lambda\text{-}defectiveness: \qquad e^{-t\lambda} P_t^U(f|_U) \geq f|_U$$

Here $e^{-t\lambda} P_t^U$ is the Dirichlet semigroup on U associated to $(-\Delta + \lambda)$: $v(t;x) = e^{-t\lambda} P_t^U f$ solves

$$\begin{cases} \partial_t v = \Delta v - \lambda v \\ v(0;x) = f(x) \text{ in the } L^1 \text{ sense} \end{cases}$$

Rmk. formally $\partial_t|_{t=0} v = \Delta f - \lambda f \geq 0$

Ex. $\lambda = 0$. $f \leq 0$ is subharmonic iff its evolution through the heat flow (w/ null boundary condition) is non-decreasing.

Well-known for regular enough f, e.g. [K.T. Sturm, Crelle, 1994],[E. Ouhabaz, PotA, 1996]

We can remove the assumption $0 \geq f$ by translating...

Say that g is λ-harmonic if $\Delta g - \lambda g = 0$.

Theorem

Let $f \in L^1_{loc}$. Then $\Delta f - \lambda f \geq 0$ in the sense of distributions iff $\forall\, U \Subset M$ there exists a λ–harmonic function g on U such that $\forall\, t > 0$ one has

$$e^{-t\lambda} P_t^U (f|_U - g) \geq f|_U - g \tag{1}$$

Def. f satisfying (1) is said *locally λ-shift defective (l.λ-s.d.).*

- One can prove directly that l.λ-s.d. f satisfies a Caccioppoli inequality. In particular $f_+^{s/2} \in W^{1,2}_{loc}$
- Once we know that f is l.λ-s.d., the monotone approximation is for free:
  - $e^{-\lambda} P_t^U$ is regularizing

  $\Rightarrow$ just take $e^{-t\lambda} P_t^U (f|_U - g) + g$ as $t = t_k \to 0$.

!!! The monotone approximation is used in the above theorem: this is not a shortcut for the BMS conjecture in the smooth case. **However...**

The latter approach suggest to take the local λ-shift defectiveness as a definition of "distributional subsolutions" of $\Delta f - \lambda f \geq 0$ (resp. subharmonic functions) also when "distributional" a priori makes no sense.

The setting: locally smoothing, strongly local Dirichlet spaces.

- locally compact, separable, metric space X with a Radon measure having a full support and $\mathcal{E}$ a strongly local, Markovian, regular Dirichlet form with dense domain of definition $\mathcal{F}(X) \subset L^2(X)$

$$\text{think:} \qquad \mathcal{E}(u,v) \approx \int < \nabla u, \nabla v >, \qquad \mathcal{F}(X) \approx W^{1,2}(X)$$

- locally smoothing: $\forall\, U \Subset X$ it holds that P_t^U is ultracontractive and $\exists U' \supset U$ open with $U' \setminus U \neq \emptyset$ such that $P_t^{U'}$ is doubly Feller and irreducible.

In this setting,

Theorem

1. *Locally λ-shift defective functions are well defined and enjoy the strong maximum principle.*
2. *If in addition X is strictly local (i.e. $d^{\mathcal{E}} \asymp d$) and complete, than it is L^p-PP for all $p \in (1, \infty)$.*

Examples (beyond smooth manifolds):

- $\mathrm{RCD}(K, N)$ spaces with $\mathcal{F}(X) = \{f : \mathrm{Ch}(f) < \infty\}$ and $\mathcal{E}$ induced by Ch.
- Carnot groups
- (only of 1.) Unbounded Sierpinski gasket

Thanks for the attention!

Markovian property: for all contractions $T : \mathbf{R} \to \mathbf{R}$ with $T(0) = 0$ and all $f \in \mathcal{F}(X)$ one has $T \circ f \in \mathcal{F}(X)$ with

$$\mathcal{E}(T \circ f, T \circ f) \leq \mathcal{E}(f, f)$$

Regularity: $\mathcal{F}(X) \cap C_c(X)$ is dense in $C_c(X)$ with respect to $\|\cdot\|_\infty$ and in $\mathcal{F}(X)$ wrt $\mathcal{E}_1(u, v) := \mathcal{E}_1(u, v) + < u, v >$

Strong locality: if u is constant on supp v then $\mathcal{E}(u, v) = 0$

Ultracontractivity: $P_t^U : L^1 \to L^\infty$

Double Feller property: $P_t^{U'} : C_0 \to C_0$ and $L^\infty \to C$ and $\lim_{t \to 0^+} P_t f = f$ pt-ws $\forall\, f \in C_0$.

$\mathbf{RCD}(\mathsf{K},\mathsf{N})$. For $f \in L^2(X)$, $\mathrm{Ch}(f) := \int_X |\nabla f|_*^2 d\mu$ where $|\nabla f|_*$ is the *minimal relaxed slope* of f, i.e. the L^2-minimal g satisfying

$$\exists\, \tilde{g} \leq g \in L^2,\ \{f_n\} \subset L^2 \text{ s.t. } f_n \xrightarrow{L^2} f,\ \mathrm{Lip}(f_n) \overset{L^2}{\rightharpoonup} f.$$

Given $f, g \in \mathcal{F}(X) := \{\mathrm{Ch} < \infty\}$ the limit

$$(\nabla f, \nabla g) = \lim_{\epsilon \to 0^+} \frac{1}{2\epsilon}(|\nabla(f + \epsilon g)|_*^2 - |\nabla f|_*^2)$$

exists in $L^1(X)$, and $\mathcal{E}(f,g) = \int (\nabla f, \nabla g)$. Finally one assumes that

- every $f \in \mathcal{F}(X)$ with $|\nabla f|_* \leq 1$ has a 1-Lipschitz μ-representative.
- $\int e^{-d(x_0,x)^2} < \infty$
- the (K,N)-Bochner inequality holds.

Carnot groups. G a simply connected Lie group with Lie algebra $\mathfrak{g}$, such that there exists $N \geq 1$ and a stratification

$$\mathfrak{g} = \mathfrak{V}_1 \oplus \cdots \oplus \mathfrak{V}_N$$

such that $[\mathfrak{V}_i, \mathfrak{V}_j] = \mathfrak{V}_{i+j}$, where $\mathfrak{V}_k := \{0\}$ for $k > N$. Let $\mathfrak{m}$ denote the Haar measure on G, and let $V_1, \ldots, V_d$ denote a basis of $\mathfrak{V}_1$, considered as left invariant vector fields on G. Then the closure $(\mathcal{E}, \mathcal{F}(G))$ in $L^2(G, \mathfrak{m})$ of the symmetric bilinear form

$$C_c^\infty(G) \times C_c^\infty(G) \ni (f, g) \longmapsto \int_G \sum_{i,j=1}^d (V_i f)(V_j g) \mathrm{d}\mathfrak{m}$$

turns $(X, \mathfrak{m}, \mathcal{E}, \mathcal{F}(G))$ into a strongly local Dirichlet space with

$$\mathcal{E}(f, g) = \int_G \sum_{i,j=1}^d (V_i f)(V_j g) \mathrm{d}\mathfrak{m} \quad \text{for all } f, g \in \mathcal{F}(G).$$

M is p-parabolic if any of the following

- some (hence every) compact set $K \subset M$ with non empty interior has p-capacity 0.
- any bounded above p-subharmonic function u is constant.
- There is no positive Green function for the p-Laplacian Δ_p on M.
- Every vector field X on M such that
  (a) $|X| \in L^{\frac{p}{p-1}}(M)$
  (b) $\operatorname{div} X \in L^1_{loc}(M)$ and $\min(\operatorname{div} X, 0) =: (\operatorname{div} X)_- \in L^1(M)$
  satisfies necessarily $0 \geq \int_M \operatorname{div} X dV_M$.
- $(p = 2)$ the Brownian motion on M is recurrent.

M is stochastically complete if any of the following

- $\int_M p_t(x, y)\, dy = 1$
- there are no non-zero bounded λ-harmonic function on M.
- $(p = 2)$ the Brownian motion has infinite lifetime: the total probability of the particle being found in the state space is constantly equal to 1.