Weighted functional inequalities for generalised Cauchy measures.

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Sobolev inequalities in the Alps

Generalised Cauchy measures $\mathbb{R}^n$. $\beta > n/2$.

$$d\mu_\beta = \frac{1}{Z_{n,\beta}}(1 + |x|^2)^{-\beta}\, dx.$$

Generalised Cauchy measures $\mathbb{R}^n$. $\beta > n/2$.

$$d\mu_\beta = \frac{1}{Z_{n,\beta}}(1+|x|^2)^{-\beta}\,dx.$$

- Up to rescaling, $\beta \to \infty$ $\mu_\beta \to \gamma$ Gaussian measure.
- Link to fast diffusion $\partial_t u = \Delta u^m$ (Barenblatt profil)

Does not satisfy Poincaré inequality...

• Bobkov-Ledoux -'09
C_β explicit such that, for all n, for all $\beta \geq n$

$$C_\beta \mathrm{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}^n} |df|(1+x^2)\, d\mu_\beta$$

Not optimal but good asymptotic as $\beta \to \infty$.

• Danzler-McCann -'05
Study of the spectrum of the adequate operator. Spectral theory.

• Bonforte-Dolbeault-Grillo-Vázquez -'10
(Hardy-)Poincaré inequality even in the non-probability case

• Nguyen -'14
Optimal constant for $\beta \geq n+1$. General result for $d\mu \sim \omega^{-\beta} dx$ and $\mathrm{Hess}(\omega) \geq \rho\,\mathrm{id}$.

$$\rho(\beta-1)\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}^n} |df|\omega\, d\mu$$

• Bonnefont-Joulin-Ma -'16
Complete dimension $n=1$ and almost dimension $n \geq 2$ except for $n + n/(n+1) \leq \beta < n+1$.

Other methods :

- Intertwining (Arnaudon-Bonnefont-Joulin - '18)
- Stein method (Saumard - '19)
- Non-negative Ricci (Gentil-Zugmeyer -'21)
- Intertwining and manifold embedding (H. - '22)

Theorem (General Cauchy measure - optimal constant)

For $n = 1$

$$\lambda_1(-L) = \begin{cases} (\beta - 1/2)^2 & \text{if} \quad 1/2 < \beta \leq 3/2 \\ 2(\beta - 1) & \text{if} \quad 3/2 \leq \beta \end{cases} .$$

For $n \geq 2$

$$\lambda_1(-L) = \begin{cases} (\beta - n/2)^2 & \text{if} \quad n/2 < \beta \leq n/2 + 2 \\ 4(\beta - n/2 - 1) & \text{if} \quad n/2 + 2 \leq \beta \leq n + 1 \\ 2(\beta - 1) & \text{if} \quad n + 1 \leq \beta \end{cases} .$$

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Secondary goals :

- error term
- extremal functions
- for general measures

M manifold. μ proba. L reversible operator. **P** semigroup associated.
Carré du champ operators :

$$2\Gamma(f) = Lf^2 - 2fLf$$
$$2\Gamma_2(f) = L\Gamma(f) - 2\Gamma(f, Lf)$$

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• $CD(\rho, \infty)$ criterion : $\Gamma_2 \geq \rho\Gamma$. Implies "Poincaré inequality"

$$\rho \mathrm{Var}_\mu(f) \leq \int_M \Gamma(f)\, d\mu$$

Example. $d\mu \sim e^{-V} d\,\mathrm{vol}$. $L = \Delta - \nabla V \cdot \nabla$. Then $\Gamma(f) = |df|^2$.
Classical Poincaré inequality.

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Classical Poincaré inequality.

Spectral gap. $CD(\rho, \infty) \Rightarrow \lambda_1(-L) \geq \rho$

$$\lambda_1(-L) = \inf\left\{ \frac{\int_M \Gamma(f)\, d\mu}{\mathrm{Var}_\mu(f)}, f \in \mathcal{D}(-L) \right\}.$$

Theorem (Integrated CD criterion)

For all $\rho > 0$*,*

$$\mathrm{Var}_\mu(f) = \frac{1}{\rho}\int_M \Gamma(f)d\mu - \frac{2}{\rho}\int_0^{+\infty}\int_M (\Gamma_2 - \rho\Gamma)(\mathbf{P}_t f)\,dt\,d\mu.$$

Moreover, $\lambda_1(-L)$ *is the largest* $\rho > 0$ *such that* $\int_M (\Gamma_2 - \rho\Gamma)\,d\mu \geq 0$.

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Moreover, $\lambda_1(-L)$ is the largest $\rho > 0$ such that $\int_M (\Gamma_2 - \rho\Gamma)\,d\mu \geq 0$.

Idea of proof. $h_t = \int_M (\mathbf{P}_t(f))^2\,d\mu$.

$$\mathrm{Var}_\mu(f) = \frac{-1}{2\rho}h'(0) - \frac{1}{\rho}\int_0^{+\infty}\frac{1}{2}h''(t) + \rho h'(t)\,dt.$$

$$\int_M \Gamma(f)\,d\mu \leq \mathrm{Var}_\mu(f)^{1/2}\left(\int_M (Lf)^2\,d\mu\right)^{1/2}.$$

Remarks :

- adaptable for Φ-entropy (log-Sobolev...)

$$h(t) = \int_M \Phi(\mathbf{P}_t f)\, d\mu.$$

- If f is extremal, then (up to regularity)

$$\int_M (\Gamma_2 - \rho\Gamma)(f)\, d\mu = 0.$$

$\omega : M \to \mathbb{R}$, smooth, $\omega > 0$. Probability $d\mu_\beta \sim \omega^{-\beta} d\,\text{vol}$.

$$L = \omega\Delta - (\beta - 1)\langle d\omega, d\cdot\rangle.$$

Carré du champ operators

$$\begin{aligned}
\Gamma(f) =& \omega|df|^2, \\
\Gamma_2(f) =& \|\omega \text{Hess}(f)\|_{HS}^2 + \omega^2 \operatorname{Ric}(\nabla f, \nabla f) \\
& + \frac{1}{2}\left[\omega\Delta\omega - (\beta - 1)|d\omega|^2\right]|df|^2 + \langle d|df|^2, \omega d\omega\rangle \\
& - \langle \Delta f df, \omega d\omega\rangle + (\beta - 1)\omega \text{Hess}(\omega)(\nabla f, \nabla f).
\end{aligned}$$

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&+ \frac{1}{2}\left[\omega\Delta\omega - (\beta-1)|d\omega|^2\right]|df|^2 + \langle d|df|^2, \omega d\omega\rangle\\
&- \langle \Delta f df, \omega d\omega\rangle + (\beta - 1)\omega \mathrm{Hess}(\omega)(\nabla f, \nabla f).
\end{aligned}$$

Idea of proof : Bochner formula

$$\frac{1}{2}\Delta|df|^2 - \langle df, d\Delta f\rangle = \|\mathrm{Hess}(f)\|_{HS}^2 + \mathrm{Ric}(\nabla f, \nabla f).$$

Do we have $CD(\rho, \infty)$?

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For generalised Cauchy measures. $\omega(x) = 1 + |x|^2$.

$$\begin{aligned}\Gamma_2(f) = {} & \|\omega \mathrm{Hess}(f) + x \otimes \nabla f + \nabla f \otimes x - \langle df, x \rangle \,\mathrm{id}\|_{HS}^2 \\ & + (n-2) \left[|df|^2 |x|^2 - \langle df, x \rangle^2 \right] \\ & + (2\beta + n - 2)|df|^2\end{aligned}$$

Hence, integrated criterion!

$$\Gamma(f) = \omega f'^2,$$
$$\Gamma_2(f) = \omega^2 f''^2 + [\omega + 2(\beta - 1)] f'^2 + 2 f'' f' x \omega.$$

Two options : factorisation or integration by parts.

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Two options : factorisation or integration by parts.

$$\int_{\mathbb{R}} \Gamma_2(f)\, d\mu_\beta = \int_{\mathbb{R}} (\omega f'' + \varepsilon x f')^2 + [A_\varepsilon + B_\varepsilon x^2] f'^2 \, d\mu_\beta.$$

- If $\beta \geq 3/2$, optimal at $\varepsilon_0 = 0$.

$$\int_{\mathbb{R}} \Gamma_2(f)\, d\mu_\beta = \int_{\mathbb{R}} \omega^2 f''^2 + 2(\beta - 1)\Gamma(f)\, d\mu_\beta.$$

besides, if f extremal, then $f'' = 0$.

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besides, if f extremal, then $f'' = 0$.

• If $1/2 < \beta \leq 3/2$, optimal at $\varepsilon_0 = 3/2 - \beta$.

$$\int_{\mathbb{R}} \Gamma_2(f)\, d\mu = \int_{\mathbb{R}} \left(\omega f'' + \left(\frac{3}{2} - \beta\right) f'\right)^2 + \left(\beta - \frac{1}{2}\right)^2 \Gamma(f) + \left(\beta - \frac{1}{2}\right)\left(\frac{3}{2} - \beta\right) f'^2\, d\mu.$$

Besides, no extremal functions, but $f_\varepsilon(x) = x\omega^\varepsilon(x)$ as $\varepsilon \uparrow (2\beta - 3)/4$.

Integration by parts

$$\int_M \langle d|df|^2, \omega d\omega\rangle \, d\mu_\beta = \int_M \left[-\omega\Delta\omega + (\beta-1)|d\omega|^2\right] |df|^2 \, d\mu_\beta.$$

$$\int_M \langle \Delta f df, \omega d\omega\rangle \, d\mu_\beta = \int_M -\frac{1}{2}\langle d|df|^2, \omega d\omega\rangle - \omega \mathrm{Hess}(\omega)(\nabla f, \nabla f) + (\beta-1)\langle df, d\omega\rangle^2 \, d\mu_\beta.$$

$$\int_M \langle d|df|^2, \omega d\omega\rangle \, d\mu_\beta = \frac{1}{\beta-2}\int_M \omega^2 \Delta|df|^2 \, d\mu_\beta.$$

$$\int_M \langle \Delta f df, \omega d\omega\rangle \, d\mu_\beta = \frac{1}{\beta-2}\int_M \omega^2\langle df, d\Delta f\rangle + (\omega\Delta f)^2 \, d\mu_\beta.$$

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$$\int_M \langle d|df|^2, \omega d\omega\rangle \, d\mu_\beta = \frac{1}{\beta-2}\int_M \omega^2 \Delta |df|^2 \, d\mu_\beta.$$

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We can get only 4 terms :

- $\|\mathrm{Hess}(f)\|^2$,
- $\|\mathrm{Hess}(f)\|_{HS}^2 - \frac{1}{n}(\Delta f)^2$,
- $|df|^2|d\omega|^2 - \langle df, d\omega\rangle^2$,
- $|df|^2$.

Strong convexity

$$\begin{aligned}\int_M \Gamma_2(f)\, d\mu_\beta = \int_M \frac{\beta-(n+1)}{\beta-2}\|\omega \mathrm{Hess}(f)\|_{HS}^2 \\ + \frac{n}{\beta-2}\left[\|\omega \mathrm{Hess}(f)\|_{HS}^2 - \frac{1}{n}(\omega\Delta f)^2\right] \\ + (\beta-1)\omega\left[\frac{1}{\beta-2}\omega\,\mathrm{Ric} + \mathrm{Hess}(\omega)\right](\nabla f, \nabla f)\, d\mu_\beta.\end{aligned}$$

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Assumption : under non-negative Ricci curvature ($\mathbb{R}^n$)

$$\mathrm{Hess}(\omega) \geq \rho_-\, \mathrm{id}\,. \tag{H1}$$

Strong convexity

Theorem

For $n \geq 2$ and $\beta \geq n+1$, under $(H1)$, we have

$$\lambda_1(-L) \geq \rho_-(\beta - 1).$$

Moreover, we have

$$\begin{aligned}
&\rho_-(\beta-1)\mathrm{Var}_{\mu_\beta}(f) - \int_M \Gamma(f)\, d\mu_\beta \\
&\quad \leq -2\int_0^{+\infty}\int_M \frac{\beta-(n+1)}{\beta-2}\|\omega \mathrm{Hess}(\mathbf{P}_t f)\|_{HS}^2 \\
&\qquad\qquad + \frac{n}{\beta-2}\left[\|\omega \mathrm{Hess}(\mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n}(\omega \Delta \mathbf{P}_t f)^2\right]\, d\mu_\beta\, dt.
\end{aligned}$$

Large β - generalised Cauchy measures

For generalised Cauchy measure, $\mathrm{Hess}(\omega) = 2\,\mathrm{id}$.

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If f is extremal, then $\|\mathrm{Hess}(f)\| = 0$ (or $n\|Hess(f)\|^2 = (\Delta f)^2$ if $\beta = n+1$).
Reciprocally, if $\beta > n/2 + 1$, linear functions are eigenfunctions associated to $2(\beta - 1)$

For $n \geq 2$ and $\beta \geq n+1$, $\lambda_1(-L) = 2(\beta - 1)$.

Bounded convexity

$$\begin{aligned}\int_M \Gamma_2(f)\, d\mu_\beta = \int_M \frac{n}{n-1}&\left[\|\omega \mathrm{Hess}(f)\|_{HS}^2 - \frac{1}{n}(\omega\Delta f)^2\right] \\ &+ \frac{(n+1-\beta)(\beta-1)}{n-1}\left[|df|^2|d\omega|^2 - \langle df, d\omega\rangle^2\right] \\ &+ \omega\left[\frac{n}{n-1}\omega\,\mathrm{Ric} + (\beta-1)\mathrm{Hess}(\omega)\right](\nabla f, \nabla f) \\ &+ \frac{n+1-\beta}{n-1}\omega\left[\mathrm{Hess}(\omega) - \Delta\omega\,\mathrm{id}\right](\nabla f, \nabla f)\, d\mu_\beta.\end{aligned}$$

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Assumption : under non-negative Ricci curvature ($\mathbb{R}^n$)

$$\rho_-\,\mathrm{id} \leq \mathrm{Hess}(\omega) \leq \rho_+\,\mathrm{id}\,. \tag{H2}$$

$$\kappa = \frac{\rho_+}{\rho_-} \geq 1.$$

Bounded convexity

Theorem

For $n \geq 2$, under $(H2)$, for all $\frac{n(n+1)\kappa-2}{n(\kappa+1)-2} < \beta \leq n+1$,

$$\lambda_1(-L) \geq \rho_- \left(\beta - 1 - \frac{n+1-\beta}{n-1}(n\kappa - 1) \right) = \tilde{c}.$$

Moreover, we have

$$\begin{aligned} &\tilde{c}\mathrm{Var}_{\mu_\beta}(f) - \int_M \Gamma(f)\, d\mu_\beta \\ &\leq -2\int_0^{+\infty} \int_M \frac{n}{n-1} \left[\|\omega \mathrm{Hess}(\mathbf{P}_t f)\|_{HS}^2 - \frac{1}{n}(\omega \Delta \mathbf{P}_t f)^2 \right] \\ &\qquad + \frac{(n+1-\beta)(\beta-1)}{n-1} \left[|d\mathbf{P}_t f|^2 |d\omega|^2 - \langle d\mathbf{P}_t f, d\omega \rangle^2 \right] \, d\mu_\beta \, dt \end{aligned}$$

Intermediate β - generalised Cauchy measures

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Reciprocally, if $\beta > n/2 + 2$, quadratic functions are eigenfunctions associated to $4(\beta - n/2 - 1)$

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$$\lambda_1(-L) = 4(\beta - n/2 - 1).$$

Small β for Cauchy

Factorisation :

$$\|\omega \text{Hess}(f)\|_{HS}^2 \rightsquigarrow \left\| \omega \text{Hess} f + \varepsilon \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|_{HS}^2$$

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$$\begin{aligned}
&\int_{\mathbb{R}^n} \Gamma_2(f)\, d\mu \\
&= \int_{\mathbb{R}^n} \mathcal{A}_\varepsilon \left[\left\| \omega \mathrm{Hess} f + \varepsilon \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|^2 - \frac{1}{n} \left(\omega \Delta f + \varepsilon \langle df, x \rangle \right)^2 \right] \\
&\quad + \mathcal{B}_\varepsilon \left[|df|^2 |x|^2 - \langle df, x \rangle^2 \right] + \left[\mathcal{C}_\varepsilon + \mathcal{D}_\varepsilon |x|^2 \right] |df|^2 \, d\mu,
\end{aligned}$$

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Small β for Cauchy

- If $\beta \geq n/2+2$, optimal at $\varepsilon_0 = 0$.
- If $n/2 < \beta \leq n/2+2$, optimal at $\varepsilon_0 = n/2+2-\beta$.

$$\int_{\mathbb{R}^n} \Gamma_2(f)\, d\mu =$$

$$\int_{\mathbb{R}^n} \frac{n}{n-1} \left[\left\| \omega \mathrm{Hess} f + \varepsilon_0 \frac{\nabla f \otimes x + x \otimes \nabla f}{2} \right\|^2 - \frac{1}{n} \left(\omega \Delta f + \varepsilon_0 \langle df, x\rangle\right)^2 \right]$$

$$+ \frac{(n-2)\left[(\beta-1)^2 - (n-2)(\beta-1) + \left(\frac{n}{2}+1\right)^2\right]}{2(n-1)} \left[|df|^2|x|^2 - \langle df, x\rangle^2\right]$$

$$+ \left(\frac{n}{2}+2-\beta\right)\left(\beta+\frac{n}{2}\right)|df|^2 + \left(\beta - \frac{n}{2}\right)^2 \Gamma(f)\, d\mu.$$

Small β for Cauchy

For all $n \geq 2$ and $\frac{n}{2} < \beta \leq \frac{n}{2} + 2$, $\lambda_1(-L) \geq \left(\beta - \frac{n}{2}\right)^2$. Moreover, we have

$$\begin{aligned}
&\left(\beta - \frac{n}{2}\right)^2 \mathrm{Var}_\mu(f) - \int_{\mathbb{R}^n} |df|^2(1+|x|^2)\, d\mu \\
&= -2\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{n}{n-1}\left\|\omega \mathrm{Hess}\mathbf{P}_t f + \varepsilon_0 \frac{\nabla \mathbf{P}_t f \otimes x + x \otimes \nabla \mathbf{P}_t f}{2}\right\|^2 \\
&\qquad - \frac{1}{n-1}\left(\omega \Delta \mathbf{P}_t f + \varepsilon_0 \langle d\mathbf{P}_t f, x\rangle\right)^2 \\
&\qquad + \frac{(n+2)(6\beta+n-4)}{4(n-1)}\left[|d\mathbf{P}_t f|^2|x|^2 - \langle d\mathbf{P}_t f, x\rangle^2\right] \\
&\qquad + \left(\frac{n}{2} + 2 - \beta\right)\left(\beta + \frac{n}{2}\right)|d\mathbf{P}_t f|^2\, d\mu\, dt.
\end{aligned}$$

where $\varepsilon_0 = \beta - \frac{n}{2} - 2$.

Small β for Cauchy

If f is extremal, then $|df| = 0$ (except in $\beta = n/2 + 2$). Then there is no extremal function.

Yet, $f_\varepsilon(x) = \omega^\varepsilon(x)$ with $\varepsilon < (2\beta - n)/4$ is in $\mathbb{L}^2(\mu_\beta)$ and

$$\lim_{\varepsilon \uparrow (2\beta - n)/4} \frac{\int_{\mathbb{R}} \Gamma(f_\varepsilon)\, d\mu_\beta}{\mathrm{Var}_{\mu_\beta}(f_\varepsilon)} = \lim_{\varepsilon \uparrow (2\beta - n)/4} \frac{\int_{\mathbb{R}} -f_\varepsilon L f_\varepsilon\, d\mu_\beta}{\int_{\mathbb{R}^n} f_\varepsilon^2\, d\mu_\beta} = \left(\beta - \frac{n}{2}\right)^2.$$

For $n \geq 2$ and $n/2 < \beta \leq n/2 + 2$, $\lambda_1(-L) = (\beta - n/2)^2$.

Goal : weighted Brascamp-Lieb inequality (see Nguyen)

$$\lambda \mathrm{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}^n} \langle \mathrm{Hess}(\omega)^{-1} \nabla f, \nabla f \rangle \omega d\mu_\beta.$$

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$$\lambda \mathrm{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}^n} \langle \mathrm{Hess}(\omega)^{-1} \nabla f, \nabla f \rangle \omega d\mu_\beta.$$

Idea :

$$Lf = \omega \nabla (\mathrm{Hess}(\omega)^{-1} \nabla f) - (\beta - 1) \langle \mathrm{Hess}(\omega)^{-1} \nabla \omega, \nabla f \rangle.$$

Then

$$\Gamma(f) = \omega \langle \mathrm{Hess}(\omega)^{-1} \nabla f, \nabla f \rangle.$$

Remarks :

- For generalised Cauchy, it is the same L as previously.
- Computation difficulties ("Bochner" formula).

dimension 1

$$Lf = \omega(af')' - (\beta - 1)a\omega' f', \quad \text{with} \quad a^{-1} = \omega''.$$

$$\int_{\mathbb{R}} \Gamma_2(f)\, d\mu_\beta = \int_{\mathbb{R}} \left(\omega(af')'\right)^2 + (\beta - 1)\Gamma(f)\, d\mu_\beta.$$

Example

$\mu \sim \mathcal{N}(0,1)$, with $\omega(x) = \exp(x^2/(2\beta))$. Then for all $\beta > 1$

$$\frac{\beta - 1}{\beta}\mathrm{Var}_\mu(f) \leq \int_{\mathbb{R}} f'^2 \frac{\beta}{x^2 + \beta}\, d\mu.$$

dimension 1 - smaller β

$$\int_{\mathbb{R}} \Gamma_2(f)\, d\mu_\beta$$
$$= \int_{\mathbb{R}} \left(\omega(af')' + \varepsilon a\omega' f'\right)^2 + \left(\beta - 1 + \varepsilon - \varepsilon(\varepsilon + \beta - 1)\frac{\omega'^2}{\omega\omega''}\right)\Gamma(f)\, d\mu_\beta.$$

Theorem

If $\frac{\omega'^2}{\omega\omega''} \leq \kappa$*, then for all* $1 - \frac{1}{\kappa} < \beta$

$$C_\beta \mathrm{Var}_{\mu_\beta}(f) \leq \int_{\mathbb{R}} f'^2 \frac{\omega}{\omega''}\, d\mu_\beta,$$

$$C_\beta = \begin{cases} (1+\kappa(\beta-1))^2/4\kappa & \textit{if} \quad 1 - \frac{1}{\kappa} < \beta \leq 1 + \frac{1}{\kappa} \\ \beta - 1 & \textit{if} \quad 1 + \frac{1}{\kappa} \leq \beta \end{cases}$$

Application - Subbotin distributions

$d\mu_\varepsilon \sim \exp(-(x^2+\varepsilon)^{\alpha/2}/\alpha)dx$ regularisation of $d\mu_\alpha \sim \exp(-|x|^\alpha/\alpha)dx$.

$$\omega_\varepsilon(x) = \exp(-(x^2+\varepsilon)^{\alpha/2}/\alpha\beta)$$

For $\beta > 0, 1 \leq \alpha \leq 2$

$$C_\beta \mathrm{Var}_{\mu_\alpha}(f) \leq \int_{\mathbb{R}} f'^2 \frac{x^{2-\alpha}}{\beta(\alpha-1)+x^\alpha}\, d\mu_\alpha$$

$$C_\beta = \begin{cases} 1/4 & \text{if} \quad 0 < \beta \leq 2 \\ (\beta-1)/\beta^2 & \text{if} \quad 2 \leq \beta \end{cases}$$

Example

$\mu \sim \exp(-|x|)$, we recover $\quad \mathrm{Var}_\mu(f) \leq 4 \int_{\mathbb{R}} f'^2\, d\mu.$

• Bobkov-Ledoux -'09
For $\beta \geq (n+1)/2$

$$(\beta - 1)\mathrm{Ent}_{\mu_\beta}(f^2) \leq \int_{\mathbb{R}^n} |df|(1+x^2)^2 \, d\mu_\beta$$

Non-optimal weight (but good asymptotic as $\beta \to \infty$).

• Cattiaux-Guillin-Wu - '11
For $\beta > n/2$. Optimal order of magnitude

$$(1+|x|^2)\log(e+|x|^2)$$

but non-explicit constant.

Integrated criterion?

$$\mathrm{Ent}_\mu(f) = \frac{1}{2\rho}\int_M f\Gamma(\log(f))\,d\mu - \frac{1}{\rho}\int_0^\infty \int_M \mathbf{P}_t f(\Gamma_2 - \rho\Gamma)(\log(\mathbf{P}_t f))\,d\mu dt.$$

$\int e^f[\Gamma_2 - \rho\Gamma](f)d\mu \geq 0$ not equivalent to Log-Sob.

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$\int e^f[\Gamma_2 - \rho\Gamma](f)d\mu \geq 0$ not equivalent to Log-Sob.

Problem. $\int e^f f'' f' \tilde{\omega}$ does not have a "nice" IPP formula.

But "bigger weight" brings "better CD-criterion".

$$\omega(x) = \sigma + |x|^2, \quad \sigma > 1.$$

$$Lf = \omega \log(\omega) \Delta f + 2[1 - (\beta - 1) \log(\omega)] \langle x, df \rangle.$$

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$$Lf = \omega \log(\omega)\Delta f + 2[1 - (\beta - 1)\log(\omega)]\langle x, df\rangle.$$

$$\Gamma_2(f) = \left\| \Omega \mathrm{Hess}(f) + \frac{\nabla\Omega \otimes \nabla f + \nabla f \otimes \nabla\Omega - \langle df, d\Omega\rangle}{2} \right\|_{HS}^2 + \mathcal{A}_\sigma \left[|df|^2|x|^2 - \langle df, x\rangle^2 \right] + [(2\beta - n)\Omega + \mathcal{R}_\sigma]|df|^2$$

with $\Omega = \omega \log(\omega)$.

Do we have $\mathcal{A}_\sigma \geq 0$ and $R_\sigma \geq 0$?

Dimension 1

$$\Gamma_2(f) = (\Omega f'' + \frac{1}{2}\Omega' f')^2 + (2\beta - 1)\omega \log(\omega) f'^2 + [x^2 + (2\beta-1)\sigma \log(\omega)^2 - 2\beta\sigma \log(\omega)] f'^2$$

$\mathcal{R}_\sigma \geq 0$ iff $\sigma \geq \sigma_0 := \exp(2\beta/(2\beta - 1))$

Theorem

For all $\beta > 1/2$, $\sigma \geq \sigma_0$, $f \in \mathcal{C}_c^\infty(\mathbb{R})$

$$(4\beta - 2)\mathrm{Ent}_{\mu_{\sigma,\beta}}(f) \leq \int_{\mathbb{R}} \frac{f'^2}{f} \omega \log(\omega) \, d\mu_{\sigma,\beta}$$

Optimality with $f_\varepsilon = \omega^\varepsilon$, $\varepsilon < \beta - 1/2$.

Dimension 1

Rescaling $x/\sqrt{\sigma} \to y$.
Optimal for σ_0 (the smallest).

Corollary

For all $\beta \geq 1/2$, for all $f \in \mathcal{C}_c^\infty(\mathbb{R})$

$$(4\beta - 2)\mathrm{Ent}_{\mu_\beta}(f) \leq \int_{\mathbb{R}} \frac{f'^2}{f}(1+x^2)\left(\frac{2\beta}{2\beta-1} + \log(1+x^2)\right) d\mu_\beta.$$

Dimension $n \geq 2$

- fails for $n = 2$...

- $n \geq 3$. $\mathcal{R}_\sigma$ negative. We need a new parameter ε

$$(2\beta - n)\Omega + \mathcal{R}_\sigma = (2\beta - n - \varepsilon)\Omega + (\mathcal{R}_\sigma + \varepsilon\Omega).$$

We obtain inequalities with non-optimal (but explicit) constant...